

Twin Vector Fields and Independence of Spectra for Quadratic Vector Fields

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Abstract The object of this paper is to address the following question: when is a polynomial vector field on \mathbb{C}^2 completely determined (up to affine equivalence) by the spectra of its singularities? We will see that for quadratic vector fields, this is not the case: given a generic quadratic vector field there is, up to affine equivalence, exactly one other vector field which has the same spectra of singularities. Let us say that two distinct vector fields are *twin vector fields* if they have the same singular locus and the same spectrum at each singularity. Our main result is as follows: any two generic quadratic vector fields with the same spectra of singularities (yet possibly different singular locus) can be transformed by suitable affine maps to be either the same vector field or a pair of twin vector fields. Moreover, a generic quadratic vector field has exactly one twin vector field. We later analyze the case of quadratic Hamiltonian vector fields in more detail and find necessary and sufficient conditions for a collection of non-zero complex numbers to arise as the spectra of singularities of a quadratic Hamiltonian vector field. Lastly, we show that a generic quadratic vector field is completely determined (up to affine equivalence) by the spectra of its singularities together with the characteristic numbers of its singular points at infinity.

Keywords Quadratic vector fields · Spectra of singularities · Euler-Jacobi formula

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1 Introduction

In this work, we will consider polynomial vector fields on the affine plane \mathbb{C}^2 . Let us denote by \mathcal{A}_n the space of all polynomial vector fields

$$v = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y},$$

such that P and Q have degree at most n . By Bezout's theorem, a generic element of \mathcal{A}_n has exactly n^2 isolated singularities; let us define then $\tilde{\mathcal{A}}_n$ to be the space of vector fields $v \in \mathcal{A}_n$ having n^2 isolated singularities. From now on, we will consider exclusively vector fields from the classes $\tilde{\mathcal{A}}_n$. Note that the space \mathcal{A}_n is naturally endowed with a structure of a complex affine space and that $\tilde{\mathcal{A}}_n$ is a Zariski open subset of \mathcal{A}_n . Throughout this work, we will say that a property is *generic* for vector fields in the class $\tilde{\mathcal{A}}_n$ if the set of vector fields having such property contains a Zariski open subset of $\tilde{\mathcal{A}}_n$. We say that two vector fields v and w are *affine equivalent* if there exists an affine map T that transforms v into w , that is, $w(x, y) = DT \cdot v(T^{-1}(x, y))$. Denote by $\text{Sing}(v)$ the singular locus of v . If $p \in \text{Sing}(v)$, we define the *spectrum* of v at p as the (unordered) pair of eigenvalues of the linearization matrix

$$Dv(p) = \left(\begin{array}{cc} P'_x & P'_y \\ Q'_x & Q'_y \end{array} \right) \Big|_{(x,y)=p}.$$

Note that if $M \in \mathfrak{gl}_2\mathbb{C}$, then the spectrum of M carries exactly the same information as the ordered pair

$$\text{Spec } M := (\text{tr} M, \det M).$$

For the remaining part of the paper, the spectrum of v at a singular point p will be thought of as the ordered pair $\text{Spec } Dv(p)$, unless otherwise specified.

If X is any topological space and $m \geq 1$, let S_m denote the symmetric group on m elements and X^m/S_m the quotient of the usual action of S_m on X^m which permutes the components. The set of spectra of singularities of a generic polynomial vector field of degree n belongs to the space

$$\mathcal{S}_n := (\mathbb{C}^2)^{n^2}/S_{n^2},$$

which is an irreducible affine algebraic variety. We have a well-defined map

$$\text{Spec}_n : \tilde{\mathcal{A}}_n \rightarrow \mathcal{S}_n.$$

In very general terms, we call *independence of spectra* the question of understanding the image and the fibers of the map Spec_n . We will see below that in the case of quadratic vector fields, we have a good understanding of this question: the closure of the image of Spec_2 is a codimension two subvariety defined by the *Euler-Jacobi relations on spectra* (i.e., Eqs. 1 and 2 in Corollary 1) and the fiber of Spec_2 over a generic point in its image consists of two disjoint orbits of the action of the affine group $\text{Aff}(2, \mathbb{C})$ on $\tilde{\mathcal{A}}_2$.

Definition 1 We say that two vector fields from the class $\tilde{\mathcal{A}}_n$ have *the same spectra of singularities* if they have the same image under the map Spec_n .

Note that the above definition involves the spectra of the singularities and does not take into account the position of these. Of particular interest are pairs of vector fields that share both position and spectra of singularities.

Definition 2 We will say that two vector fields $v_1, v_2 \in \tilde{\mathcal{A}}_n$ are *twin vector fields* if they are not equal, yet they have exactly the same singular locus and, for each point p in the common singular set, the matrices $Dv_1(p)$ and $Dv_2(p)$ have the same spectrum.

The next two theorems are the main results of this work.

Theorem 1 *If two quadratic vector fields on \mathbb{C}^2 with non-degenerate singularities have the same spectra of singularities then, possibly after transforming one of them by a suitable affine map, they either agree or they are twin vector fields. Moreover, a generic quadratic vector field has exactly one twin vector field.*

Note that in the generic case, a pair of quadratic twin vector fields cannot be affine equivalent to each other; hence, the above theorem implies that given a generic vector field v , there exist exactly two disjoint orbits of the action of the affine group on $\tilde{\mathcal{A}}_2$ consisting of vector fields with the same spectra as v .

Theorem 2 *A generic quadratic vector field is completely determined (up to affine equivalence) by the spectra of its finite singularities and the characteristic numbers of its singular points at infinity.*

An important question about independence of spectra is the *realization problem*:

Question 1 *Which collections of numbers in \mathcal{S}_n can be realized as the spectra of a degree n polynomial vector field?*

We shall answer this question in the particular case of quadratic Hamiltonian vector fields in Theorem 5.

We remark that Theorem 1 is very similar in spirit to results in [4] (for foliations on \mathbb{P}^2 of degree two) and [3] (for foliations on \mathbb{P}^2 coming from a generic quadratic vector field on \mathbb{C}^2), where it is proved that in the generic case the *Baum-Bott indices* completely determine a foliation up to finite ambiguity (modulo the natural action of $\mathrm{PGL}(2, \mathbb{C})$ and $\mathrm{Aff}(2, \mathbb{C})$, respectively). In fact, Lins Neto proves that the generic fiber of the Baum-Bott map, which associates to a foliation the Baum-Bott indices of its singularities, contains exactly 240 orbits of the natural action of $\mathrm{PGL}(2, \mathbb{C})$.

This paper is organized as follows: we shall first discuss a few general facts about polynomial vector fields before specializing to the case of quadratic ones and proving Theorem 1. After this, we will analyze the case of quadratic Hamiltonian vector fields. We then discuss Theorem 2 and conclude with a related open question.

2 The Euler-Jacobi Formula

Let us recall the statement of the Euler-Jacobi formula in the particular case relevant to us [1, Chpt. 5, Sec. 2].

Theorem 3 *If P, Q are polynomials in $\mathbb{C}[x, y]$ of degree n whose divisors intersect transversely in n^2 different points $p_1, \dots, p_{n^2} \in \mathbb{C}^2$ and $g(x, y)$ is a polynomial of degree at most $2n - 3$ then*

$$\sum_{k=1}^{n^2} \frac{g(p_k)}{\mathbf{J}(p_k)} = 0,$$

where $\mathbf{J}(x, y)$ is the Jacobian determinant $\mathbf{J}(x, y) = \det \frac{\partial(P, Q)}{\partial(x, y)}$.

Consider a polynomial vector field $v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ of degree $n \geq 2$. By making $g(x, y) = 1$ or $g(x, y) = \text{tr}(Dv(x, y))$ we obtain polynomials whose value at the singular point p_k depends exclusively on the spectrum of $Dv(p_k)$.

Corollary 1 *If $v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ is a polynomial vector field of degree $n \geq 2$ having n^2 non-degenerate singularities $p_1, \dots, p_{n^2} \in \mathbb{C}^2$ then*

$$\sum_{k=1}^{n^2} \frac{1}{\det Dv(p_k)} = 0, \tag{1}$$

$$\sum_{k=1}^{n^2} \frac{\text{tr}(Dv(p_k))}{\det Dv(p_k)} = 0. \tag{2}$$

We call these equations the *Euler-Jacobi relations on spectra*.

3 Counting Dimensions

Let us exclude the trivial case $n = 1$ in the following discussion. The space \mathcal{A}_n has dimension $(n + 1)(n + 2)$. The affine group $\text{Aff}(2, \mathbb{C})$ acts on \mathcal{A}_n in a natural way and it is clear that affine equivalent vector fields have the same spectra of singularities. Let us define $\widehat{\mathcal{A}}_n = \widetilde{\mathcal{A}}_n / \text{Aff}(2, \mathbb{C})$, that is, the quotient of this action in the sense of geometric invariant theory; we have

$$\dim \widehat{\mathcal{A}}_n = n^2 + 3n - 4.$$

On the other hand, the map Spec_n was defined to have codomain $\mathcal{S}_n := (\mathbb{C}^2)^{n^2} / S_{n^2}$, which has dimension $2n^2$. Let us denote by $\widehat{\mathcal{S}}_n$ the Zariski closure of the set of points in \mathcal{S}_n with non-zero components that satisfy (1) and (2). In this way Spec_n actually takes values on $\widehat{\mathcal{S}}_n$. We have

$$\dim \widehat{\mathcal{S}}_n = 2n^2 - 2.$$

We obtain an induced map on the quotient $\widehat{\text{Spec}}_n : \widehat{\mathcal{A}}_n \rightarrow \widehat{\mathcal{S}}_n$, and the above arguments show that $\dim \widehat{\mathcal{S}}_n \geq \dim \widehat{\mathcal{A}}_n$, with equality only when $n = 2$. The case $n = 2$, where both dimensions are equal, is analyzed in the next section; we will see that, modulo affine equivalence, the spectra determines the vector field up to finite ambiguity (in fact, up to its unique twin). This can be rephrased by saying that the generic fiber of the induced map

$$\widehat{\text{Spec}}_2 : \widehat{\mathcal{A}}_2 \rightarrow \widehat{\mathcal{S}}_2$$

consists of two points — or equivalently, that it is a generically two-to-one map.

Because of the above dimension count, we see that for $n > 2$, not every element of $\widehat{\mathcal{S}}_n$ lies in the closure of the image of $\widehat{\text{Spec}}_n$, and so, the spectra of singularities of vector fields from $\widetilde{\mathcal{A}}_n$ must satisfy additional relations besides (1) and (2). Whether or not the map $\widehat{\text{Spec}}_n$ is injective when $n > 2$ is unknown to the author.

4 Quadratic Vector Fields

The next lemma is an essential step in the proof of Theorem 1.

Lemma 1 *Let v be a quadratic vector field having four non-degenerate singularities p_1, \dots, p_4 . The position and spectrum of p_4 is completely determined by the position and spectra of p_1, p_2, p_3 .*

Proof This lemma is a double application of the Euler-Jacobi formula. First, we can think of relations (1) and (2) as a system of equations which we can solve for $\text{Spec } Dv(p_4)$ every time we are given $\text{Spec } Dv(p_k)$ for $k = 1, 2, 3$. Second, if we let $g_1(x, y) = x$ and $g_2(x, y) = y$ in the Euler-Jacobi formula, then the system of equations

$$\sum_{k=1}^4 \frac{g_1(p_k)}{\det Dv(p_k)} = 0, \quad \sum_{k=1}^4 \frac{g_2(p_k)}{\det Dv(p_k)} = 0,$$

determines completely the position of p_4 . □

Note that the above lemma implies, in particular, that two quadratic vector fields are twins if and only if three out of their four singularities agree in position and spectra.

Remark 1 It is straightforward to show that if a polynomial vector field on \mathbb{C}^2 of degree n has $n + 1$ collinear singularities then the whole line through these points is singular itself. In particular, a vector field from the class $\widetilde{\mathcal{A}}_2$ cannot have three collinear singularities. This fact will be used in the next proof and throughout this work.

Proof of Theorem 1 Suppose that two quadratic vector fields v, \tilde{v} with non-degenerate singularities have the same spectra. After transforming \tilde{v} by a suitable affine map on \mathbb{C}^2 we may assume that v and \tilde{v} have three singularities that agree in position and spectra. By Lemma 1, the same holds for the fourth singularity. We conclude that either $\tilde{v} = v$ or \tilde{v} is a twin vector field of v .

We now prove that twin vector fields exist and are unique. Consider a quadratic vector field $v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ having four non-degenerate singularities p_1, \dots, p_4 . By Max Noether’s theorem, any quadratic polynomial H which vanishes on the singular set $\text{Sing}(v) = \{P = 0\} \cap \{Q = 0\}$ can be written uniquely as

$$H = \alpha P + \beta Q,$$

for some complex numbers α, β . This means that any quadratic vector field that vanishes on the singular set $\text{Sing}(v)$ can be uniquely written as

$$\tilde{v} = (aP + bQ) \frac{\partial}{\partial x} + (cP + dQ) \frac{\partial}{\partial y},$$

for complex numbers a, b, c, d . Let us denote by A the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and note that $D\tilde{v}(x, y) = A \cdot Dv(x, y)$. In virtue of Lemma 1, the vector field \tilde{v} has the same spectra as v if and only if they have the same spectra at p_1, p_2, p_3 . This happens if and only if

$$\begin{aligned} \operatorname{tr}(A \cdot Dv(p_k)) &= \operatorname{tr}(Dv(p_k)), \quad \text{for } k = 1, 2, 3, \\ \det A &= 1. \end{aligned} \quad (3)$$

The above is a system of three linear equations and one quadratic equation on a, b, c, d . If the system is independent, we can always eliminate three of these variables using the linear equations and then substitute into the quadratic one. This gives a quadratic equation in one variable which generically would have two different solutions. In order to check that for a generic quadratic vector field, the linear system is independent and the discriminant of the quadratic equation is not zero; it is enough to find a single example with these properties. A simple example is given by the Hamiltonian vector field

$$v = x(x + 2y - 1) \frac{\partial}{\partial x} + y(-2x - y + 1) \frac{\partial}{\partial y}. \quad (4)$$

The computations are straightforward and we will omit them here.

This proves that, for a generic quadratic vector field, system (3) has two solutions: one corresponds to the original vector field v and the other to a different vector field \tilde{v} , thus establishing existence and uniqueness of a twin vector field for a generic vector field v . \square

5 Quadratic Hamiltonian Vector Fields

The quadratic Hamiltonian case can be very clearly understood. Let \mathcal{H}_2 denote the space of quadratic Hamiltonian vector fields. It is straightforward that this space has dimension 9, since the space of polynomials in $\mathbb{C}[x, y]$ of degree three is ten dimensional. Moreover, a vector field on \mathbb{C}^2 is Hamiltonian if and only if the trace $\operatorname{tr}(Dv(x, y)) = P'_x + Q'_y$ vanishes identically. This imposes three linear conditions on the coefficients of P and Q , and so \mathcal{H}_2 is in fact a linear subspace of \mathcal{A}_2 .

Remark 2 Note that $\operatorname{tr}(Dv(x, y))$, which is a linear polynomial on (x, y) , will vanish identically as soon as it vanishes on three non-collinear points. This implies that a vector field on \mathcal{A}_2 is Hamiltonian if and only if it has a vanishing trace at each of its singularities. This implies, moreover, any vector field that is affine equivalent to a Hamiltonian vector field with isolated singularities is itself Hamiltonian.

The quotient $\mathcal{H}_2 // \operatorname{Aff}(2, \mathbb{C})$ has dimension 3. On the other hand, each singularity of a Hamiltonian vector field carries only one analytic invariant: the determinant of its linearization matrix. These invariants are subject to the Euler-Jacobi relation (1) and so the space of possible spectra is three-dimensional. Once again, the dimension of the space of essential parameters matches the dimension of the space of invariants. Because the space of quadratic Hamiltonian vector fields \mathcal{H}_2 is a linear (and hence irreducible) subspace of \mathcal{A}_2 , the fact that the system (3) is non-degenerate at the Hamiltonian vector field (4) implies not

only that a generic quadratic vector field has a unique twin but also that a *generic* quadratic Hamiltonian vector field has a unique twin.

However, in the case of quadratic Hamiltonian vector fields we can prove the existence and uniqueness of twin vector fields in a different way, and in this way, we can explicitly say what our genericity assumptions are.

Definition 3 Let v be a quadratic Hamiltonian vector field whose singularities are all non-degenerate. We say that the spectrum of v is *exceptional* if there exist singular points p_j, p_k such that

$$\det Dv(p_j) + \det Dv(p_k) = 0.$$

A quadratic Hamiltonian vector field is a *generic quadratic Hamiltonian vector field* if it has non-degenerate singularities and its spectrum is not exceptional.

Note that if $\det Dv(p_j) + \det Dv(p_k) = 0$, then the eigenvalues of $Dv(p_j)$ differ from the eigenvalues of $Dv(p_k)$ only by multiplication of a common factor of $\sqrt{-1}$. We now have the following theorem.

Theorem 4 A generic quadratic Hamiltonian vector field has a unique twin vector field. In fact, the twin vector field of a generic quadratic Hamiltonian vector field v is precisely its negative $-v$.

Before proving the above theorem, let us point out that besides making the genericity assumptions explicit in the Hamiltonian case, we will also be able to solve the realization problem.

Definition 4 A collection of four non-zero complex numbers $\mathcal{C} = \{d_1, \dots, d_4\}$ is called *admissible* for quadratic Hamiltonian vector fields if it satisfies

$$\sum_{k=1}^4 \frac{1}{d_k} = 0.$$

An admissible collection is called *exceptional* if there exist d_j, d_k such that $d_j + d_k = 0$.

Theorem 5 A collection \mathcal{C} , admissible for quadratic Hamiltonian vector fields, is realizable as the spectrum of a quadratic Hamiltonian vector field if and only if one of the following two conditions holds:

1. \mathcal{C} is a non-exceptional collection,
2. \mathcal{C} is exceptional of the form $\{d, -d, d, -d\}$, for some $d \in \mathbb{C}^*$.

Proof of Theorems 4 and 5 Every quadratic vector field with four isolated singularities is affine equivalent to a vector field with singularities at $p_1 = (0, 0), p_2 = (1, 0), p_3 = (0, 1)$. Any such vector field is completely determined by six parameters. Indeed, if $v = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ vanishes on such points then P and Q must be of the form

$$\begin{aligned} P(x, y) &= a_1x^2 + a_2xy + a_3y^2 - a_1x - a_3y, \\ Q(x, y) &= a_4x^2 + a_5xy + a_6y^2 - a_4x - a_6y, \end{aligned} \tag{5}$$

for arbitrary $a_1, \dots, a_6 \in \mathbb{C}$. In the particular case in which the vector field in question is Hamiltonian we further obtain the following:

$$\begin{aligned} a_2 &= 2a_1, \\ a_5 &= -2a_1, \\ a_6 &= -a_1. \end{aligned}$$

Should read: A short computation shows that d_k , the determinant of $Dv(p_k)$, is given by

$$\begin{aligned} d_1 &= -a_1^2 - a_3a_4, \\ d_2 &= -a_1^2 - 2a_1a_4 + a_3a_4, \\ d_3 &= -a_1^2 + 2a_1a_3 + a_3a_4. \end{aligned} \quad (6)$$

If the spectrum of v is non-exceptional we can always solve the above system of equations for a_1, a_3, a_4 in terms of d_1, d_2, d_3 to obtain:

$$\begin{aligned} a_1^2 &= \frac{-(d_1 + d_2)(d_1 + d_3)}{2(d_2 + d_3)}, \\ a_3 &= a_1 + \frac{d_1 + d_3}{2a_1}, \\ a_4 &= -a_1 - \frac{d_1 + d_2}{2a_1}. \end{aligned}$$

This proves that non-exceptional admissible collections are realizable. Note that these expressions yield two solutions (depending on the branch of the square root chosen for a_1^2) and one solution differs from the other by a sign, hence proving Theorem 4.

Now, assume v is a Hamiltonian vector field with exceptional spectrum; without loss of generality we can assume $d_1 + d_2 = 0$. Note that the Euler-Jacobi relation (1) implies that $d_3 + d_4 = 0$. From Eq. 6 we deduce that either $a_1 = 0$ or $a_1 + a_4 = 0$. If $a_1 = 0$ then we conclude that $d_3 = d_2$ (and thus $d_4 = d_1$) and if $a_1 + a_4 = 0$ we conclude that $d_3 = d_1$ (and thus $d_4 = d_2$). In either case we see that exceptional spectra of quadratic Hamiltonian vector fields are always of the form $\mathcal{C} = \{d, -d, d, -d\}$, for some $d \in \mathbb{C}^*$. On the other hand any such collection \mathcal{C} can be realized (choosing $d_3 = d_2$) by setting $a_1 = 0$ and a_3, a_4 any complex numbers that satisfy $a_3a_4 = d$, since Eq. 6 becomes

$$\begin{aligned} d_1 &= -a_3a_4, \\ d_2 &= a_3a_4, \\ d_3 &= a_3a_4. \end{aligned} \quad \square$$

Note that the last argument in the above proof shows that, given an exceptional collection $\mathcal{C} = \{d, -d, d, -d\}$, there exists a one-dimensional family of quadratic Hamiltonian vector fields which are pairwise not affine equivalent and realize \mathcal{C} as their spectra of singularities. This implies that a quadratic Hamiltonian vector field with non-degenerate singularities and exceptional spectrum has a one-dimensional family of twin vector fields.

6 Characteristic Numbers at Infinity

It follows from Theorem 1 that generic quadratic vector fields are determined by their spectra of singularities up to finite ambiguity. This suggests that if we take into account additional analytic invariants we should expect to have enough information to single out a unique vector field. Thus far, we have not yet considered the singular points at infinity.

For these singular points we cannot define unambiguously their spectra but we can always define their characteristic numbers. There is, however, a slight difficulty that we will find; suppose we are given polynomials P, Q defining a vector field, in order to compute the characteristic numbers at infinity, we must first *find* the singular points at infinity; this involves solving a cubic equation. In order to bypass this, we will work backwards: we will assume that we are given the position and characteristic numbers of the singularities at infinity and use this information to recover the polynomials P and Q (subject to the normalization given by requiring the vector field to have singularities at $p_1 = (0, 0), p_2 = (1, 0), p_3 = (0, 1)$).

The following proposition is a well-known fact and its proof is immediate.

Proposition 1 *In the generic case, there is a one-to-one correspondence between quadratic homogeneous vector fields*

$$\xi(x, y) = P_2 \frac{\partial}{\partial x} + Q_2 \frac{\partial}{\partial y},$$

modulo multiplication by a non-zero complex number and foliations with linear monodromy

$$\frac{dz}{dw} = z \sum_{j=1}^3 \frac{\mu_j}{w - w_j}, \quad \mu_1 + \mu_2 + \mu_3 = 1,$$

defined in a neighborhood of the line at infinity $z = 0$ (where z, w are the coordinates $(z, w) = (1/x, y/x)$).

In virtue of the above proposition, we can uniquely define a quadratic homogeneous vector field by specifying three points $w_1, w_2, w_3 \in \mathbb{P}^1$, three numbers $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$ satisfying $\mu_1 + \mu_2 + \mu_3 = 1$, and a rescaling factor $\kappa \in \mathbb{C}^*$. Once such homogeneous vector field is given, we use formula (5) to define uniquely a quadratic vector field having singularities at $p_1 = (0, 0), p_2 = (1, 0), p_3 = (0, 1)$.

A straightforward computation provides the following explicit formulae:

$$\begin{aligned} a_1 &= \kappa (\mu_1 w_2 w_3 + \mu_2 w_1 w_3 + \mu_3 w_1 w_2), \\ a_2 &= -\kappa (\mu_1 (w_2 + w_3) + \mu_2 (w_1 + w_3) + \mu_3 (w_1 + w_2)), \\ a_3 &= \kappa, \\ a_4 &= \kappa w_1 w_2 w_3, \\ a_5 &= -\kappa (\mu_1 w_1 (w_2 + w_3) + \mu_2 w_2 (w_1 + w_3) + \mu_3 w_3 (w_1 + w_2)), \\ a_6 &= \kappa (\mu_1 w_1 + \mu_2 w_2 + \mu_3 w_3). \end{aligned}$$

In order to prove that a generic vector field is completely determined by the spectra of its finite singularities and the characteristic numbers at infinity we will define a *moduli map* \mathcal{M} that assigns to each vector field its set of spectra and characteristic numbers.

In virtue of Lemma 1, the spectra of v is completely determined by the spectra at p_1, p_2, p_3 . Let us define a map $\text{Spec}: \mathbb{C}^6 \rightarrow \mathbb{C}^6$ that assigns to each vector field as above its spectra of singularities at p_1, p_2, p_3 . Namely,

$$\text{Spec}(a_1, \dots, a_6) = (\text{Spec } p_1, \text{Spec } p_2, \text{Spec } p_3). \tag{7}$$

Note that the map Spec is polynomial and its components are either linear (for the traces) or quadratic (for the determinants).

Theorem 1 may now be restated as follows.

Theorem 1' *The map $\text{Spec}: \mathbb{C}^6 \rightarrow \mathbb{C}^6$ is a regular dominant map whose generic fiber consists of two points.*

We now define the moduli map $\mathcal{M}: \mathbb{C}^6 \rightarrow \mathbb{C}^2 \times \mathbb{C}^6$ by the formula

$$\mathcal{M}(\mu_1, \mu_2, w_1, w_2, w_3, \kappa) = ((\mu_1, \mu_2), \text{Spec}(a_1, \dots, a_6)).$$

In order to prove Theorem 2, we need to show that the generic fiber of the moduli map \mathcal{M} consists of a single point. In order to do this, it is enough to find a non-empty open set $W \subset \text{Im}\mathcal{M}$ and $U \subset \mathbb{C}^6$ such that $U = \mathcal{M}^{-1}(W)$ and $\mathcal{M}|_U: U \rightarrow W$ is one-to-one.

Proof of Theorem 2 Consider the following vector field

$$v = (x^2 + 2xy - x) \frac{\partial}{\partial x} + (-xy + 3y^2 - 3y) \frac{\partial}{\partial y}.$$

A simple computation shows that the unique twin vector field of v is

$$\tilde{v} = (3x^2 + 6xy - 3x) \frac{\partial}{\partial x} + \left(-\frac{7}{3}x^2 - 5xy + y^2 + \frac{7}{3}x - y \right) \frac{\partial}{\partial y},$$

and that the derivative of the map Spec is invertible both at v and at \tilde{v} . Let $\Lambda = \text{Spec}v$. We can deduce from the inverse function theorem and from Theorem 1' the existence of neighborhoods W_1, U, \tilde{U} of Λ, v, \tilde{v} , respectively, such that $\text{Spec}^{-1}(W_1) = U \cup \tilde{U}$ and Spec maps both U and \tilde{U} diffeomorphically onto W_1 . On the other hand, it is not hard to check that the characteristic numbers at infinity of v and \tilde{v} are different (the characteristic numbers of v are all rational whereas the characteristic numbers of \tilde{v} are not). By shrinking U and \tilde{U} if necessary we can assume that vector fields in U have different characteristic numbers from any vector field in \tilde{U} . This means that if (μ_1, μ_2) are the characteristic numbers of v we can find a small neighborhood W_0 of (μ_1, μ_2) such that no vector field from \tilde{U} has characteristic numbers in W_0 . Define $W = W_0 \times W_1 \subset \mathbb{C}^2 \times \mathbb{C}^6$. Since $\mathcal{M} = (\mu_1, \mu_2, \text{Spec})$ and $\text{Spec}^{-1}(W_1) = U \cup \tilde{U}$, we must have $\mathcal{M}^{-1}(W) \subset U \cup \tilde{U}$. However, we also know that $\mathcal{M}^{-1}(W)$ is disjoint from \tilde{U} , by construction of W_0 , and so $\mathcal{M}^{-1}(W) = U$. We conclude that \mathcal{M} is one-to-one over W , and so, the generic fiber consists of a single point. \square

7 The image of the moduli map \mathcal{M}

As shown by Theorem 2, the closure of the image of the moduli map $\mathcal{M}: \mathbb{C}^6 \rightarrow \mathbb{C}^8$ is a codimension two subvariety of \mathbb{C}^8 . This implies that there are at least two independent relations between the spectra of finite singularities and the characteristic numbers at infinity. One of these is the well-known *Baum-Bott formula*, yet there must exist one more relation.

Question 2 *What is the missing relation between the spectra of finite singularities and the characteristic numbers at infinity for a generic quadratic vector field?*

The above question is closely related to a question on the *hidden relations* between the spectra of the derivatives at the fixed points of a regular endomorphism $f: \mathbb{P}^2 \rightarrow \mathbb{P}^2$, posed by Adolfo Guillot in [2]. There are known relations (generalizing the *holomorphic Lefschetz fixed point formula*) among these eigenvalues yet a dimensional argument shows that there must be even more relations. Guillot's question is: what are those missing relations?

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