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The utmost rigidity property for quadratic foliations on $\mathbb{P}^2$ with an invariant line

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Abstract In this work, we consider holomorphic foliations of degree two on the complex projective plane $\mathbb{P}^2$ having an invariant line. In a suitable choice of affine coordinates, these foliations are induced by a quadratic vector field over the affine part in such a way that the invariant line corresponds to the line at infinity. We say that two such foliations are topologically equivalent provided there exists a homeomorphism of $\mathbb{P}^2$ which brings the leaves of one foliation onto the leaves of the other and preserves orientation both on the ambient space and on the leaves. The main result of this paper is that in the generic case, two such foliations may be topologically equivalent if and only if they are analytically equivalent. In fact, it is shown that the analytic conjugacy class of the holonomy group of the invariant line is the modulus of both topological and analytic classification. We obtain as a corollary that two generic orbitally topologically equivalent quadratic vector fields on $\mathbb{C}^2$ must be orbitally affine equivalent. This result improves, in the case of quadratic foliations, a well-known result by Ilyashenko that claims that two generic and topologically equivalent foliations with an invariant line at infinity are affine equivalent, provided they are close enough in the space of foliations and the linking homeomorphism is close enough to the identity map of $\mathbb{P}^2$.

Keywords Holomorphic foliations · Topological rigidity · Holonomy group at infinity

Mathematics Subject Classification Primary 37F75; Secondary 32M25 · 34M35

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1 Introduction

Any polynomial vector field on $\mathbb{C}^2$ with isolated singularities defines a singular holomorphic foliation by curves which can be analytically extended to the projective plane $\mathbb{P}^2$. Conversely, any holomorphic foliation on $\mathbb{P}^2$ with isolated singularities is given by a polynomial vector field on any affine chart. We are interested in foliations on $\mathbb{P}^2$ with an invariant line. It is convenient to choose affine coordinates, such that the invariant line becomes the line at infinity. Since any line can be mapped to any other line by a linear automorphism of $\mathbb{P}^2$, there is no loss of generality in choosing a distinguished line $\mathscr{L}$ and considering only foliations which leave $\mathscr{L}$ invariant. Define $\mathscr{A}_n$ to be the class of those singular foliations on $\mathbb{P}^2$ which in the fixed affine chart $\mathbb{C}^2 \approx \mathbb{P}^2 \backslash \mathscr{L}$ are induced by a polynomial vector field of degree n and have an invariant line at infinity. Note that the line at infinity with the singularities removed is a leaf of the foliation. We call this leaf the leaf at infinity or the infinite leaf indistinctly.

Remark 1.1 Foliations from the class $\mathscr{A}_n$ have, by definition, *affine degree n*, since they are induced by a polynomial vector field on $\mathbb{C}^2$ of degree n. The fact that they have an invariant line at infinity implies that such foliations also have *projective degree n*. By projective degree n, we mean that such foliations have exactly n tangencies with any line not invariant by the foliation (cf. [1,2]).

Two foliations from the class $\mathscr{A}_n$ are *topologically equivalent* if there exists an orientation-preserving homeomorphism of $\mathbb{P}^2$ that brings the leaves of the first foliation onto the leaves of the second one and preserves the natural orientation on these leaves. In case such a map is an affine map on $\mathbb{C}^2$, we say that the foliations are affine equivalent.

Let $\mathscr{F} \in \mathscr{A}_n$ and denote by $\mathscr{L}_{\mathscr{F}}$ its leaf at infinity. Given a base point $b \in \mathscr{L}_{\mathscr{F}}$, the germ of a cross section (Γ, b) transversal to the leaves of $\mathscr{F}$ and a parametrization $(\mathbb{C}, 0) \to (\Gamma, b)$ we obtain the holonomy representation $\Delta\colon \pi_1(\mathscr{L}_{\mathscr{F}}, b) \to \mathrm{Diff}(\mathbb{C}, 0)$ of the fundamental group of the infinite leaf on the space $\mathrm{Diff}(\mathbb{C}, 0)$ of germs of invertible holomorphic maps at $(\mathbb{C}, 0)$. Its image is called the *holonomy group at infinity* of $\mathscr{F}$.

Definition 1.1 We say that two foliations $\mathscr{F}$ and $\widetilde{\mathscr{F}}$ from the class $\mathscr{A}_n$ have *analytically conjugate holonomy groups at infinity* whenever there exist the germ of a conformal map $h \in \mathrm{Diff}(\mathbb{C}, 0)$ and a geometric isomorphism[1] $H_*\colon \pi_1(\mathscr{L}_{\mathscr{F}}, b) \to \pi_1(\mathscr{L}_{\widetilde{\mathscr{F}}}, \tilde{b})$, such that for any loop $\gamma \in \pi_1(\mathscr{L}_{\mathscr{F}}, b)$ we have $h \circ \Delta_\gamma = \widetilde{\Delta}_{H_*\gamma} \circ h$.

1.1 Rigidity of polynomial foliations

Generic foliations from the class $\mathscr{A}_n$ exhibit a phenomenon known as topological rigidity. Topological rigidity of polynomial foliations was, until now, more a heuristic idea than a formal statement. The idea of topological rigidity is that topological equivalence of foliations implies their analytic equivalence. There are several theorems in the literature asserting that topological equivalence of generic foliations plus some additional hypotheses implies their affine equivalence. The first such rigidity property for generic polynomial foliations was discovered by Ilyashenko in [3] and called *absolute rigidity*.

Definition 1.2 We say that a foliation $\mathscr{F} \in \mathscr{A}_n$ is *absolutely rigid* if there exist a neighborhood U of $\mathscr{F}$ in $\mathscr{A}_n$ and a neighborhood V of the identity map in the space of self-homeomorphisms of $\mathbb{P}^2$, such that any foliation from U which is conjugate to $\mathscr{F}$ by a homeomorphism in V is necessarily affine equivalent to $\mathscr{F}$.

It is proved in [3] that a generic polynomial foliation is absolutely rigid. However, their genericity assumptions excluded a dense subset of $\mathscr{A}_n$. These assumptions have been substantially weakened by Shcherbakov Nakai and others (cf. [5,7,12]). In the latest works, the key assumption on a foliation is the non-solvability of its holonomy group at infinity.

Later on, Ilyashenko and Moldavskis proved that generic quadratic foliations exhibit a stronger rigidity property, known as *total rigidity* [4].

Definition 1.3 A polynomial foliation $\mathscr{F} \in \mathscr{A}_n$ is *totally rigid* if there exist only a finite number of foliations (up to affine equivalence) from the class $\mathscr{A}_n$ which are topologically equivalent to $\mathscr{F}$.

In [4], the number of affine classes of foliations which are topologically equivalent to a given generic foliation from $\mathscr{A}_2$ is estimated to be at most 240. This result is

[1] We say that the isomorphism $H_*\colon \pi_1(\mathscr{L}_{\mathscr{F}}, b) \to \pi_1(\mathscr{L}_{\widetilde{\mathscr{F}}}, \tilde{b})$ is geometric if it is induced by some orientation-preserving homeomorphism $H\colon \mathscr{L}_{\mathscr{F}} \to \mathscr{L}_{\widetilde{\mathscr{F}}}$.

proved using the topological invariance of the Baum–Bott indices under transversely holomorphic topological equivalences.

In this work, we prove for the first time that the paradigm of topological rigidity of polynomial foliations may be formalized, at least in the case of quadratic foliations with an invariant line: two generic foliations from $\mathscr{A}_2$ are topologically equivalent if and only if they are affine equivalent—no additional assumption is made. Moreover, this is proved by comparing the holonomy groups at infinity exclusively and we thus conclude that it is the holonomy group that serves as a modulus of analytic (hence, also topological) classification.

1.2 Statement of the theorem

The following theorem is the main result of this work.

Theorem 1.1 *Let $\mathscr{F} \in \mathscr{A}_2$ be a generic foliation and suppose its holonomy group at infinity is analytically conjugate to the holonomy group of $\widetilde{\mathscr{F}} \in \mathscr{A}_2$. There exists an affine map on $\mathbb{C}^2$ that conjugates $\mathscr{F}$ to $\widetilde{\mathscr{F}}$.*

It is well known that generic topologically equivalent foliations have analytically conjugate holonomy groups [3]. The next results follows immediately from Theorem 1.1.

Corollary 1.1 *Two generic foliations from $\mathscr{A}_2$ are topologically equivalent if and only if they are affine equivalent.*

We say that two vector fields are *orbitally topologically equivalent* whenever there exists an orientation-preserving homeomorphism of $\mathbb{C}^2$ that maps the integral curves of the first vector field onto those of the second one. If two quadratic vector fields on $\mathbb{C}^2$ are orbitally topologically equivalent, it need not be true that the induced foliations on $\mathbb{P}^2$ are topologically equivalent, since the linking homeomorphism need not extend to the line at infinity. However, if the singularities at infinity are hyperbolic, it can be proved that such linking homeomorphism takes the separatrix set of the former foliation onto the separatrix set of the latter one (cf. [13]). Once this has been established, we may carry out with no problem an argument by Marín which guarantees that, even though the homeomorphism need not extend to the infinite line, the holonomy groups at infinity are still conjugated (see Theorem A in [6]). We obtain the following result.

Corollary 1.2 *Two generic quadratic vector fields on $\mathbb{C}^2$ are orbitally topologically equivalent if and only if they are orbitally affine equivalent.*

The above results may be summarized as follows.

Corollary 1.3 *Let $\mathscr{F}, \widetilde{\mathscr{F}} \in \mathscr{A}_2$ be generic foliations. The following are equivalent.*

1. *There exists a homeomorphism of $\mathbb{C}^2$ conjugating $\mathscr{F}$ to $\widetilde{\mathscr{F}}$.*
2. *There exists a homeomorphism of $\mathbb{P}^2$ conjugating $\mathscr{F}$ to $\widetilde{\mathscr{F}}$.*
3. *Foliations $\mathscr{F}$ and $\widetilde{\mathscr{F}}$ have analytically equivalent holonomy groups at infinity.*
4. *There exists an affine map on $\mathbb{C}^2$ conjugating $\mathscr{F}$ to $\widetilde{\mathscr{F}}$.*

1.3 Genericity assumptions

To prove Theorem 1.1, we shall consider exclusively foliations from the class $\mathscr{A}_2$ that satisfy the generic properties listed below.

- (i) The holonomy group at infinity is non-solvable.
- (ii) The characteristic numbers $\lambda_1, \lambda_2, \lambda_3$ of the singular points at infinity are pairwise different and do not belong to the set $\frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z}$.
- (iii) The commutator of the two holonomy maps corresponding to the *standard geometric generators*[2] of the fundamental group of the infinite leaf belongs to the class of parabolic germs with non-zero quadratic term (see Remark 2.1 in Sect. 2.1).

Moreover, there is an additional technical requirement needed to prove Theorem 1.1. In Sect. 3.3, we shall construct a dense Zariski open set $\mathscr{U} \subset \mathscr{A}_2$ and assume

- (iv) foliation $\mathscr{F}$ belongs to the set $\mathscr{U}$;

to prove Corollaries 1.2 and 1.3, we must further assume that the characteristic numbers λ_1, λ_2, λ_3 are non-real (i.e., the singularities on the line at infinity are hyperbolic). However, this last condition is not needed to prove Theorem 1.1.

The genericity of conditions (ii) and (iv) is obvious. Condition (iii) also defines a complex Zariski open set in $\mathscr{A}_2$ (cf. [12]). The genericity of (i) is proved in [12] for polynomial foliations of arbitrary degree. For quadratic vector fields, we know an even stronger result:

Theorem 1.2 [9] *Let* $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$ *be such that* $\lambda_1 + \lambda_2 + \lambda_3 = 1$. *Denote by* $\mathscr{B}_\Lambda$ *the set of foliations in* $\mathscr{A}_2$ *with characteristic numbers at infinity* $\lambda_1, \lambda_2, \lambda_3$. *Assume that* $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \operatorname{Re}\lambda_3$. *Then, if* $\lambda_1, \lambda_2 \notin \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z}$, *there exist at least one and at most ten orbits of the group* $\operatorname{Aff}(2, \mathbb{C})$ *in* $\mathscr{B}_\Lambda$ *whose points correspond to equations with non-commutative solvable holonomy group at infinity.*

Moreover, for any Λ, foliations in $\mathscr{B}_\Lambda$ with commutative holonomy group at infinity fall into seven families which are explicitly described in [9]. In particular, it follows from such description (see also Theorem 1 in [8]) that for $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfying assumption (ii) above, there exist exactly two orbits of the group $\operatorname{Aff}(2, \mathbb{C})$ in $\mathscr{B}_\Lambda$ corresponding to equations with a commutative holonomy group.

Throughout this text, we will also assume that we have once and for all numbered the singular points at infinity of any given foliation in such a way that $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \operatorname{Re}\lambda_3$.

[2] The standard geometric generators μ_i are described in Definition 2.1.

2 Structure of the work

2.1 Ideas behind the proof of Theorem 1.1

Any foliation $\mathscr{F} \in \mathscr{A}_2$ is induced in a neighborhood of the line at infinity $\{z = 0\}$, by a rational differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{z\,P(z, w)}{Q(z, w)}, \tag{2.1}$$

such that $Q|_{z=0}$ is not identically zero. In fact, the roots of $r(w) = Q(0, w)$ determine the position of the singular points at infinity which from now on will be assumed, without loss of generality, to be given by $w_1 = -1$, $w_2 = 1$ and $w_3 = \infty$. Under this assumption, the polynomial $r(w) := Q(0, w)$ may be chosen to be $r(w) = w^2 - 1$.

In Sect. 4, we will normalize the above equation using the action of the group $\mathrm{Aff}(2, \mathbb{C})$. This normalization was originally introduced in [8]. Any normalized foliation is uniquely defined by five complex parameters: the characteristic numbers λ_1, λ_2 and three more parameters $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$. We will write $\mathscr{F} = \mathscr{F}(\lambda, \alpha)$ whenever we wish to emphasize that $\mathscr{F}$ is defined by the parameters $\lambda = (\lambda_1, \lambda_2)$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2)$.

Let us also consider the solution $\Phi(z, w)$ of Eq. (2.1) with initial condition $\Phi(z, 0) = z$ and expand it as a power series in z using the variations φ_d of the solution $z = 0$ in the following way:

$$\Phi(z, w) = \sum_{d=1}^{\infty} \varphi_d(w)\, z^d.$$

The variations $\varphi_d(w)$ are defined in a neighborhood of the origin and can be analytically continued along any path on $\mathscr{L}_{\mathscr{F}}$. Moreover, the holonomy map $\Delta_\gamma(z)$ with respect to a given loop $\gamma \in \pi_1(\mathscr{L}_{\mathscr{F}}, 0)$ is given by the power series

$$\Delta_\gamma(z) = \varphi_{1\{\gamma\}}(0)\, z + \varphi_{2\{\gamma\}}(0)\, z^2 + \cdots, \tag{2.2}$$

where $\varphi_{d\{\gamma\}}$ denotes the analytic continuation of φ_d along the curve γ.

Note that the fundamental group of the leaf $\mathscr{L}_{\mathscr{F}} \cong \mathbb{C}\backslash\{-1, 1\}$ is a free group on two generators.

Definition 2.1 Let μ_1 and μ_2 be loops in $\mathscr{L}_{\mathscr{F}}$ based at the origin which go around the singular points $w = -1$ and $w = 1$, respectively, once in the positive direction. We call these loops the *standard geometric generators* of $\pi_1(\mathscr{L}_{\mathscr{F}}, 0)$.

Now, consider the commutators

$$\gamma_1 = \mu_2\mu_1\mu_2^{-1}\mu_1^{-1} \quad \text{and} \quad \gamma_2 = \mu_2\mu_1^2\mu_2^{-1}\mu_1^{-2}, \tag{2.3}$$

and let f_1, f_2 be the holonomy maps corresponding to the above loops, that is, $f_j = \Delta_{\gamma_j}, j = 1, 2$. We call this germs *distinguished parabolic germs*; they play a key role in this paper.

Remark 2.1 Genericity assumption (iii) in Sect. 1.3 can be translated to requiring that the distinguished parabolic germ

$$f_1 = [\Delta_{\mu_1}, \Delta_{\mu_2}]$$

has a non-zero quadratic term.

Suppose $\tilde{\mathscr{F}} \in \mathscr{A}_2$ is topologically equivalent to $\mathscr{F}(\lambda, \alpha)$. The genericity assumptions imposed on these foliations imply that both $\mathscr{F}$ and $\tilde{\mathscr{F}}$ have the same characteristic numbers at infinity and so we may write $\tilde{\mathscr{F}} = \mathscr{F}(\lambda, \beta)$, where β is some triple of complex numbers $\beta = (\beta_0, \beta_1, \beta_3)$. Define $\tilde{f}_j$ to be the holonomy map of $\tilde{\mathscr{F}}$ along γ_j. The topological conjugacy gives rise to a conformal germ $h \in \mathrm{Diff}(\mathbb{C}, 0)$ and a geometric automorphism H_* of $\pi_1(\mathscr{L}_{\mathscr{F}}, 0)$ which conjugate the holonomy groups as in Definition 1.1.

Remark 2.2 It follows from [10] that the geometric automorphism H_* may always be assumed to be the identity map. We therefore conclude the existence of a germ $h \in \mathrm{Diff}(\mathbb{C}, 0)$ such that

$$h \circ f_j - \tilde{f}_j \circ h = 0, \qquad j = 1, 2. \tag{2.4}$$

Because of the above, from now on we will always assume that any given analytic conjugacy between holonomy groups is given by some germ $h \in \mathrm{Diff}(\mathbb{C}, 0)$ and the identity automorphism of the fundamental group of $\mathscr{L}_{\mathscr{F}}$. In [10] such a conjugacy is called *strong analytic equivalence*. However, since we will always assume $H_* = id$, we shall not use this term.

The essence of the proof of Theorem 1.1 may be summarized as follows: If the holonomy groups of $\mathscr{F}$ and $\tilde{\mathscr{F}}$ are analytically conjugate, then there exits $h \in \mathrm{Diff}(\mathbb{C}, 0)$ such that (2.4) holds. We can compute the first terms in the power series expansions of the distinguished parabolic germs in terms of the parameters λ, α and β as explicit iterated integrals using the variation equations of the differential equation (2.1) with respect to the solution $z = 0$. We also expand h as a power series with unknown coefficients and substitute all these series into Eq. (2.4) to obtain an expression of the form

$$h \circ f_j - \tilde{f}_j \circ h = \sum_{d=1}^{\infty} \kappa_{d,j}\, z^d,$$

for $j = 1$ and $j = 2$. Equating each $\kappa_{d,j}$ to zero should impose some conditions on the parameter β. However, since we do not know the coefficients in the power series expansion of h, we must consider, for each d, the system of equations

$$\kappa_{d,1} = 0, \qquad \kappa_{d,2} = 0. \tag{2.5}$$

A careful analysis of such a system will allow us to compute the coefficient of degree $d-1$ in the power series of h and at the same time obtain a concrete condition imposed on the parameter β by (2.5). We do this for $d = 3, 4, 5, 6$. We will first obtain the conditions imposed on β expressed in terms of the vanishing of certain integrals. Even though these conditions are polynomial in β, the coefficient of such polynomials are transcendental functions on λ and α. A crucial step in the proof of Theorem 1.1 is that we are actually able to translate these conditions into algebraic ones. This is done using a Lemma 2.2, which is proved in [8]. We lastly prove that for generic λ and α, the polynomial system of equations we obtain has a unique solution given by $\beta = \alpha$. This proves that these normalized foliations having conjugate holonomy groups are in fact one and the same. This shows in particular that two foliations, not necessarily normalized, with conjugate holonomy groups must be affine equivalent. Moreover, to obtain such affine map, taking one foliation into the other we consider first the affine maps taking each foliation to its normal form and compose one of these maps with the inverse of the other.

The proof outlined above is carried out in a series of lemmas whose formal statements are given below.

2.2 Three fundamental lemmas

The most elaborate part of the proof of Theorem 1.1 is to obtain explicit conditions imposed on β by the conjugacy of the holonomy groups of $\mathcal{F}(\lambda, \alpha)$ and $\mathcal{F}(\lambda, \beta)$. We do this following closely the constructions presented in [9].

Key Lemma *For* $d = 3, 4, 5, 6$ *there exists a polynomial* $P_d(w)$*, whose coefficients are polynomials in* β*, such that the existence of a germ* $h \in \mathrm{Diff}(\mathbb{C}, 0)$ *that conjugates the holonomy groups of* $\mathcal{F}(\lambda, \alpha)$ *and* $\mathcal{F}(\lambda, \beta)$ *up to jets of order d implies*

$$\int_{\gamma_1} \frac{P_d(w)}{r(w)^d}\, \varphi_1(w)^{d-1}\, \mathrm{d}w = 0. \tag{2.6}$$

In the lemma above, $\varphi_1(w)$ is the first variation of the solution $z = 0$ of Eq. (2.1) and $r(w) = w^2 - 1$. Before proving this lemma, it is necessary to obtain explicit expressions for the coefficients in the power series expansions of the distinguished parabolic germs. These computations are carried out in Sect. 5.

Remark 2.3 Note that the vanishing of the integral in the key lemma imposes one linear condition on the coefficients of the polynomial $P_d(w)$. The polynomials $P_d(w)$ *do* depend on the foliation $\mathcal{F}(\lambda, \alpha)$. In fact, the coefficients of these polynomials depend polynomially on α and rationally on λ. The main content of the next lemma is that, in virtue of Lemma 2.2, the linear condition imposed on the coefficients of P_d by the vanishing of the integral is not trivial. This implies right away that such condition is a *polynomial* condition on the parameters β. This is discussed in detail in Sect. 3.2.

Main Lemma *For* $d = 3, 4, 5, 6$ *there exists a non-zero polynomial* $F_d \in \mathbb{C}[\beta]$, *such that the existence of a germ* $h \in \mathrm{Diff}(\mathbb{C}, 0)$ *that conjugates the holonomy groups of* $\mathcal{F}(\lambda, \alpha)$ *and* $\mathcal{F}(\lambda, \beta)$ *up to jets of order d implies* $F_d(\beta) = 0$.

Suppose now that $\mathscr{F}(\lambda, \alpha)$ and $\mathscr{F}(\lambda, \beta)$ have conjugate holonomy groups. The above lemma implies that $\beta \in \mathbb{C}^3$ satisfies the polynomial system of equations

$$F_3(\beta) = 0, \ldots, F_6(\beta) = 0. \tag{2.7}$$

This is a system of four equations on three variables. Generically, such a system will have no solutions at all. However, because of the defining property of F_d, we see that $\beta = \alpha$ will always be a solution. The proof of Theorem 1.1 is completed by the following lemma.

Elimination Lemma *There exists a non-empty Zariski open set* $U \subset \mathbb{C}^5$ *such that if* $(\lambda, \alpha) \in U$, *then the polynomial system* (2.7) *has a unique solution given by* $\beta = \alpha$.

2.3 Two lemmas about integrals

The following lemmas were proved and used by Pyartli in [8] and [9]. They play a major role in our proof and will be used frequently.

Recall that γ_1 and γ_2 have been defined to be the commutators $\gamma_1 = \mu_2\mu_1\mu_2^{-1}\mu_1^{-1}$ and $\gamma_2 = \mu_2\mu_1^2\mu_2^{-1}\mu_1^{-2}$, where μ_1, μ_2 are standard geometric generators of the fundamental group of the punctured line $\mathbb{C}\backslash\{1, -1\}$.

Lemma 2.1 *Let* $P(w)$ *be a polynomial and let* $\zeta(w) = (1+w)^{u_1}(1-w)^{u_2}$ *where* u_1, u_2 *are complex numbers and* $\zeta(0) = 1$. *Then,*

$$\int_{\gamma_2} P(w)\zeta(w)\,\mathrm{d}w = (1 + \exp(2\pi i\, u_1)) \int_{\gamma_1} P(w)\zeta(w)\,\mathrm{d}w.$$

The proof of this lemma is straightforward: we decompose the loops γ_1, γ_2 into pieces and write down each integral as a sum of integrals along these pieces to verify that the equality holds.

The next lemma is the fundamental step for deducing the main lemma from the key lemma.

Lemma 2.2 *Let* $\zeta(w) = (1+w)^{u_1}(1-w)^{u_2}$, $\zeta(0) = 1$, $u_1, u_2 \notin \mathbb{Z}$, $r(w) = w^2 - 1$ *and* $P(w)$ *a polynomial of degree at most* m. *The equality* $\int_{\gamma_1} P(w)\zeta(w)\,\mathrm{d}w = 0$ *holds if and only if there exists a constant* $C \in \mathbb{C}$ *and a polynomial* $R(w)$ *of degree at most* $\max(m-1, -2-\mathrm{Re}(u_1+u_2))$, *such that*

$$\int_0^w P(t)\zeta(t)\,\mathrm{d}t = R(w)r(w)\zeta(w) + C.$$

In this paper, we will only use the above lemma in situations where the inequality $m - 1 > -2 - \mathrm{Re}(u_1 + u_2)$ holds, so that, if it exists, $R_d(w)$ will have degree at most $m-1$. Note that both the vanishing of the integral and the existence of $R(w)$ impose one non-trivial linear condition on the coefficients of the polynomial $P(w)$. Clearly, the existence of such an R implies the vanishing of the integral, since we are integrating along the commutator loop γ_1 and so $\zeta_{\{\gamma_1\}}(0) = \zeta(0) = 1$. This implies that both linear conditions are equivalent. A detailed proof can be found in [8] and the essence of the proof is discussed in Sect. 3.2.

Recall that we have numbered the singular points at infinity of $\mathscr{F}$ in such a way that $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \operatorname{Re}\lambda_3$. It follows from the fact that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and

$$\operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2 \geq 2/3. \tag{2.8}$$

This remark will be frequently used as a complement to Lemma 2.2. In Sect. 6, we will apply Lemma 2.2 to integrals of the form (2.6) taking $u_i = (d-1)\lambda_i - d$, for $d = 3, 4, 5, 6$. To use Lemma 2.2, we require $u_i \notin \mathbb{Z}$. This is one of the instances where it is important that the genericity assumption $\lambda_i \notin \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z}$ holds.

3 Sketch of the proofs

3.1 Key lemma: the strategy

Suppose there exists a germ $h \in \operatorname{Diff}(\mathbb{C}, 0)$ that conjugates the holonomy groups of $\mathscr{F} = \mathscr{F}(\lambda, \alpha)$ and $\tilde{\mathscr{F}} = \mathscr{F}(\lambda, \beta)$. We expand the distinguished parabolic germs in power series

$$f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \cdots, \quad \tilde{f}_j(z) = z + \tilde{a}_{2j}z^2 + \tilde{a}_{3j}z^3 + \cdots, \quad j = 1, 2, \tag{3.1}$$

as well as the germ h,

$$h(z) = h_1 z + h_2 z^2 + h_3 z^3 + \cdots.$$

Note that the first variations satisfy $\varphi_1 = \tilde{\varphi}_1$, since these functions are completely determined by λ. Throughout this work, we will omit the tilde on $\tilde{\varphi}_1$.

The coefficients a_{dj} are computed in Sect. 5 in terms of the parameters λ and α. In particular, it will be shown that

$$a_{2j} = \tilde{a}_{2j} = \int_{\gamma_j} \frac{1}{r(t)}\,\varphi_1(t)\,\mathrm{d}t, \quad j = 1, 2. \tag{3.2}$$

The key lemma for degree $d = 3$ will be easily deduced from the fact that Eq. (3.2) holds, which in turn is a direct consequence of the particular normal form (4.1) that we shall be using. Furthermore, it will be shown that the equality $a_{2j} = \tilde{a}_{2j}$ forces the germ h to be parabolic; that is, $h_1 = 1$. The key lemma for all higher degrees is proved following a strategy which we now present.

Suppose we have computed all the coefficients $h_2, \ldots, h_{d-2}$ in terms of λ, α, β. Since the germs f_j, $\tilde{f}_j$ and h are parabolic, the coefficient of degree d in the power series expansion of $h \circ f_j - \tilde{f}_j \circ h$ is of the form

$$\frac{1}{d!}(h \circ f_j - \tilde{f}_j \circ h)^{(d)}(0) = (h_d + a_{dj}) - (\tilde{a}_{dj} + h_d) + \cdots = a_{dj} - \tilde{a}_{dj} + \cdots, \tag{3.3}$$

where the multiple dots denote those terms that depend only on a_{kj}, $\tilde{a}_{kj}$ and h_k with $k < d$. Since $h \circ f_j - \tilde{f}_j \circ h = 0$, the above equation yields an expression for $\tilde{a}_{dj} - a_{dj}$ in terms of a_{kj}, $\tilde{a}_{kj}$ and h_k with $k = 2, \ldots, d-1$. On the other hand, we have explicit formulas for the coefficients a_{dj}, and thus for $\tilde{a}_{dj} - a_{dj}$, from Sect. 5 (cf. Propositions 5.1 to 5.5). We equate this formula for $\tilde{a}_{dj} - a_{dj}$ to the formula we deduced from (3.3). This method yields an equation involving the index j and thus by making $j = 1$ and $j = 2$, we obtain a system of two equations. A priori, it is not at all clear what conditions this system of equations imposes on the parameter β. The fundamental fact about this system, proved in Sect. 6, is that it can be simplified to take the form

$$a_{2j}\,\mathscr{C}_d + \mathscr{I}_{dj} = 0, \quad j = 1, 2,$$

where a_{2j} is as in (3.2), $\mathscr{C}_d$ is an expression involving the coefficients $h_2, \ldots, h_{d-1}$ that does not depend on the index j, $\mathscr{I}_{dj} = \int_{\gamma_j} \frac{P_d}{r^d}\,\varphi_1^{d-1}\,\mathrm{d}w$, and P_d is a polynomial which will be computed explicitly. The key lemma for degree d is completed by the following proposition.

Proposition 3.1 *Let $d \geq 3$. If $\lambda_1 \notin \frac{1}{d-2}\mathbb{Z}$ and the polynomial $P_d(w)$ satisfies a system of equations of the form*

$$\begin{aligned} a_{21}\,\mathscr{C}_d + \mathscr{I}_{d1} &= 0 \\ a_{22}\,\mathscr{C}_d + \mathscr{I}_{d2} &= 0, \end{aligned} \tag{3.4}$$

where $\mathscr{C}_d$ is a complex number,

$$\mathscr{I}_{dj} = \int_{\gamma_j} \frac{P_d(w)}{r(w)^d}\,\varphi_1(w)^{d-1}\,\mathrm{d}w, \tag{3.5}$$

and a_{2j} is as in (3.2), then

$$\mathscr{C}_d = \mathscr{I}_{dj} = 0.$$

Proof We can regard (3.4) as a linear system on three unknowns: $\mathscr{C}_d$, $\mathscr{I}_{d1}$ and $\mathscr{I}_{d2}$. Note that the integrand that appears in (3.5) can be rewritten as $P_d(w)\zeta_d(w)$, where

$$\zeta_d(w) = \frac{1}{r(w)^d}\varphi_1(w)^{d-1} = (1+w)^{(d-1)\lambda_1 - d}(1-w)^{(d-1)\lambda_2 - d},$$

since $\varphi_1(w) = (1+w)^{\lambda_1}(1-w)^{\lambda_2}$ (cf. expression (4.4) and the variation equation (4.3)). Applying Lemma 2.1, we can express $\mathscr{I}_{d2}$ as a scalar multiple of $\mathscr{I}_{d1}$,

$$\mathscr{I}_{d2} = (1 + \nu_1^{d-1})\,\mathscr{I}_{d1}, \quad \nu_1 = \exp(2\pi i\,\lambda_1).$$

Since a_{2j} is given in terms of the integral in (3.2), Lemma 2.1 also implies that

$$a_{22} = (1 + \nu_1)\,a_{21}.$$

In this way, system (3.4) becomes

$$\begin{aligned} a_{21}\,\mathscr{C}_d + \mathscr{I}_{d1} &= 0, \\ (1+\nu_1)\,a_{21}\,\mathscr{C}_d + (1+\nu_1^{d-1})\,\mathscr{I}_{d1} &= 0, \end{aligned} \tag{3.6}$$

whose unknowns are $\mathscr{C}_d$ and $\mathscr{I}_{d1}$. The determinant of this linear system is

$$\begin{vmatrix} a_{21} & 1 \\ (1+\nu_1)\,a_{21} & 1+\nu_1^{d-1} \end{vmatrix} = a_{21}\nu_1(\nu_1^{d-2}-1),$$

which is not zero. Indeed, $\nu_1^{d-2} \neq 1$ since $\nu_1 = \exp(2\pi i\,\lambda_1)$ and $\lambda_1 \notin \frac{1}{d-2}\mathbb{Z}$, and by our genericity assumptions $a_{21} \neq 0$. This implies that $\mathscr{I}_{d1} = 0$ and $\mathscr{C}_d = 0$. □

Note that the fact $\mathscr{I}_{d1} = 0$ proves the key lemma for degree d, since the expression for $\mathscr{I}_d$ given in (3.5) coincides with the left hand side of (2.6) in the key lemma. On the other hand, $\mathscr{C}_d$ is given in terms of $h_2, \ldots h_{d-1}$ and so the fact that $\mathscr{C}_d = 0$ allows us to find an expression for the coefficient h_{d-1}. In this way, we are able to repeat the process now for degree $d+1$. That is, at every step d we will prove the key lemma for degree d and compute h_{d-1}.

3.2 Deducing main lemma from key lemma

As pointed out in Remark 2.3, the equation

$$\int_{\gamma_1} \frac{P_d}{r^d}\varphi_1^{d-1}\,\mathrm{d}w = 0 \tag{3.7}$$

imposes one linear condition on the coefficients of the polynomial $P_d(w)$. Since these coefficients are polynomials on β, we need only prove that this linear condition is non-trivial to conclude the main lemma. We prove this fact using Lemma 2.2. Indeed, Lemma 2.2 claims that Eq. (3.7) is equivalent to the existence of a polynomial $R_d(w)$ such that

$$\int_0^w \frac{P_d}{r^d}\varphi_1^{d-1}\,\mathrm{d}t = \frac{R_d(w)}{r(w)^{d-1}}\,\varphi_1(w)^{d-1} + C.$$

This means that

$$\left(\frac{R_d(w)}{r(w)^{d-1}}\,\varphi_1(w)^{d-1}\right)' = \frac{P_d(w)}{r(w)^d}\,\varphi_1(w)^{d-1},$$

on the other hand a short computation shows that

$$\left(\frac{R_d(w)}{r(w)^{d-1}}\,\varphi_1(w)^{d-1}\right)' = \frac{R_d'(w)r(w) + (d-1)(s(w)-r'(w))R_d(w)}{r(w)^d}\,\varphi_1(w)^{d-1},$$

where $s(w) = \lambda_1(w-1) + \lambda_2(w+1)$ and we have taken into account the fact that φ_1 satisfies the variation equation (4.3). This implies that

$$P_d = R_d' r + (d-1)(s - r')R_d. \tag{3.8}$$

We will see in Sect. 7.1 that the polynomials P_d have degree $2(d-1)$ and that $\deg R_d \leq \deg P_d - 1$. This fact, together with Eq. (3.8), implies that the linear condition imposed on the coefficients of $P_d(w)$ by Eq. (3.7) is non-trivial. The main lemma now follows immediately.

Remark 3.1 In Sect. 7.2, we will explain how to obtain explicit expressions for the polynomials $R_d(w)$ and $F_d(\beta)$ in terms of the coefficients of the polynomials $P_d(w)$. These will be later needed to prove the elimination lemma.

3.3 The elimination lemma

The last step in the proof of Theorem 1.1 is to prove that the system,

$$F_3(\beta) = 0, \ \ldots, \ F_6(\beta) = 0,$$

has no solutions other than $\beta = \alpha$. This is done taking resultants of the polynomials F_d with respect to successive variables $\beta_2, \beta_1, \beta_0$. Consider for the time being the parameters λ, α to be fixed; thus the coefficients of the polynomials F_d are also fixed complex numbers.

Recall that if $f(x) = a_0 x^n + \cdots + a_n$ and $g(x) = b_0 x^m + \cdots + b_m$ are polynomials in x with coefficients in some field $\mathbb{F}$, the resultant of f and g is defined to be

$$\mathrm{Res}_x(f(x), g(x)) = a_0^m b_0^n \prod_{i,j} (u_i - v_j),$$

where u_i and v_j are the roots of $f(x)$ and $g(x)$, respectively, in $\overline{\mathbb{F}}$. The resultant can be defined for polynomials over any commutative ring. Over an integral domain, it has the fundamental property that $\mathrm{Res}_x(f(x), g(x)) = 0$ if and only if $f(x)$ and $g(x)$ have a common factor of positive degree.

We will first take several resultants of the polynomials F_d with respect to β_2. Second, we take resultants of these previously obtained resultants with respect to β_1. The final step has a twist; if we take now a last resultant with respect to β_0, we are guaranteed to get 0, since $\beta = \alpha$ is a solution to system (2.7). We avoid this by dividing one of these resultants by the linear polynomial $\beta_0 - \alpha_0$. More precisely, let us define

$$\begin{aligned}
\mathrm{Res}_j^1(\beta_0, \beta_1) &= \mathrm{Res}_{\beta_2}\left(F_3(\beta_0, \beta_1, \beta_2), F_j(\beta_0, \beta_1, \beta_2)\right), && j = 4, 5, 6, \\
\mathrm{Res}_j^2(\beta_0) &= \mathrm{Res}_{\beta_1}\left(\mathrm{Res}_4^1(\beta_0, \beta_1), \mathrm{Res}_j^1(\beta_0, \beta_1)\right), && j = 5, 6, \\
\mathrm{Res}_6^3 &= \mathrm{Res}_{\beta_0}\left(\mathrm{Res}_5^2(\beta_0)/(\beta_0 - \alpha_0), \mathrm{Res}_6^2(\beta_0)\right).
\end{aligned}$$

Note that as long as we fix α and λ, we have that

$$\mathrm{Res}_j^1 \in \mathbb{C}[\beta_0, \beta_1], \quad \mathrm{Res}_j^2 \in \mathbb{C}[\beta_0], \quad \mathrm{Res}_6^3 \in \mathbb{C}.$$

Proposition 3.2 *If* $\mathrm{Res}_6^3 \neq 0$, *then any solution* (u_0, u_1, u_2) *of the polynomial system* (2.7) *satisfies* $u_0 = \alpha_0$.

Proof Suppose on the contrary that $\mathrm{Res}_6^3 \neq 0$, but (u_0, u_1, u_2) is a solution of (2.7) such that $u_0 \neq \alpha_0$. Note that $F_3(u_0, u_1, \beta_2)$ and $F_j(u_0, u_1, \beta_2)$ have a common root $\beta_2 = u_2$ for any $j = 4, 5, 6$ and so

$$0 = \mathrm{Res}_{\beta_2}\big(F_3(u_0, u_1, \beta_2), F_j(u_0, u_1, \beta_2)\big) = \mathrm{Res}_j^1(u_0, u_1), \qquad j = 4, 5, 6.$$

In particular, $\mathrm{Res}_4^1(u_0, \beta_1)$ has a common root, $\beta_1 = u_1$, with both $\mathrm{Res}_5^1(u_0, \beta_1)$ and $\mathrm{Res}_6^1(u_0, \beta_1)$. We deduce that $\mathrm{Res}_5^2(u_0) = 0$ and $\mathrm{Res}_6^2(u_0) = 0$. Now, since $u_0 \neq \alpha_0$, it is still true that $\mathrm{Res}_5^2(\beta_0)/(\beta_0 - \alpha_0)$ and $\mathrm{Res}_6^2(\beta_0)$ have $\beta_0 = u_0$ as a common root; in particular, $\mathrm{Res}_6^3 = 0$, a contradiction. □

We would like to be able to guarantee that Res_6^3 is never zero, no matter the choice of λ and α. This need not be true. However, we can guarantee that for *almost every* choice of λ and α, the resultant Res_6^3 is not zero. Indeed, as mentioned in Remark 2.3, the coefficients of the polynomials F_d depend polynomially on α and rationally on λ. In this way, if we allow α and λ to vary, the coefficients of F_d belong to the ring $\mathbb{C}(\lambda)[\alpha]$, in particular $\mathrm{Res}_6^3 \in \mathbb{C}(\lambda)[\alpha]$. Let us thus introduce the notation $\mathrm{Res}_6^3(\lambda, \alpha)$. If $\mathrm{Res}_6^3(\lambda, \alpha)$ is not identically zero, then the union of its divisors of zeroes and poles defines a proper algebraic subset of affine space $\mathbb{C}^5$. The complement U of this algebraic set is a Zariski-open subset of $\mathbb{C}^5$ with the property that for any $(\lambda, \alpha) \in U$, we have $\mathrm{Res}_6^3(\lambda, \alpha) \neq 0$. Finally, we will prove that $\mathrm{Res}_6^3(\lambda, \alpha) \not\equiv 0$ by exhibiting an explicit point $(\lambda, \alpha) \in \mathbb{C}^5$, given in (7.3), for which Res_6^3 does not vanish.

The above argument shows that if $\mathscr{F} = \mathscr{F}(\lambda, \alpha)$ and $\widetilde{\mathscr{F}} = \mathscr{F}(\lambda, \beta)$ have conjugate monodromy groups, then we must have $\alpha_0 = \beta_0$. The polynomial $F_3(\beta)$ is linear and $F_4(\beta)$ is linear on β_1, β_2, yet quadratic on β_0. However, if we replace β_0 by α_0, we obtain a linear system on β_1, β_2 (this is verified by direct inspection of the polynomials F_3 and F_4 whose explicit expression can be found in the appendix of [11]). The proof of the elimination lemma is completed by the following proposition.

Proposition 3.3 *The pair of equations*

$$F_3(\alpha_0, \beta_1, \beta_2) = 0, \qquad F_4(\alpha_0, \beta_1, \beta_2) = 0, \tag{3.9}$$

forms a linear inhomogeneous system on β_1 *and* β_2. *Its determinant is a non-zero element of* $\mathbb{C}(\lambda)[\alpha]$ *and therefore for almost every* $(\lambda, \alpha) \in \mathbb{C}^5$ *the system has a unique solution which is necessarily given by*

$$\beta_1 = \alpha_1, \qquad \beta_2 = \alpha_2.$$

The proof of this proposition is discussed in Sect. 7.3. Propositions 3.2 and 3.3 together imply the elimination lemma.

Remark 3.2 In the proof of the Main lemma and key lemma, all computations are carried out in terms of the rational functions $K_d(w)$ defined by the formula

$$\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{z\,P(z,w)}{Q(z,w)} = \sum_{d=1}^{\infty} K_d(w)\,z^d,$$

whose explicit dependence on (λ, α) is not provided until Sect. 7. This has been done to avoid excessively large expressions and make the proof more transparent. However, to prove the elimination lemma (more precisely, that the final resultant Res_6^3 and the determinant of (3.9) do not vanish identically), we do need to compute expressions for the polynomials F_d in terms of the parameters λ, α, β explicitly. Obtaining these expressions and evaluating the resultant Res_6^3 and the determinant of (3.9) at a particular point have been done with computer assistance. This procedure is discussed in Sect. 7 and the program script can be found in the appendix of [11].

4 Definitions and normalizations

A foliation $\mathscr{F} \in \mathscr{A}_2$ has three singular points at infinity. These can be brought to any other three different points on the infinite line by the action of the affine group of $\mathbb{C}^2$. We wish to normalize a foliation in such a way that the singular points are given by $w_1 = -1$, $w_2 = 1$ and $w_3 = \infty$ in coordinates $(z, w) = (1/x,\ y/x)$. If the characteristic numbers are pairwise different, we can do this unambiguously by numbering the singular points in such a way that $\mathrm{Re}\,\lambda_1 \geq \mathrm{Re}\,\lambda_2 \geq \mathrm{Re}\,\lambda_3$ and if $\mathrm{Re}\,\lambda_i = \mathrm{Re}\,\lambda_j$ then $\mathrm{Im}\,\lambda_i > \mathrm{Im}\,\lambda_j$ provided $i < j$.

Since the characteristic numbers are not integer numbers, it follows from [8] that we can find an affine change of coordinates such that in the chart (z, w) the foliation is induced by

$$\frac{\mathrm{d}z}{\mathrm{d}w} = z\,\frac{s(w)(1+\alpha_0 z) + \kappa z + \eta z^2}{r(w)(1+\alpha_0\sigma z) + p(w) z^2},$$

where $r(w) = w^2 - 1$, $s(w) = \lambda_1(w-1) + \lambda_2(w+1)$, $p(w) = \alpha_1(w-1) + \alpha_2(w+1)$, $\sigma = \lambda_1 + \lambda_2$ and $\eta = \alpha_1 + \alpha_2$.

It follows from [9] that if $\lambda_1, \lambda_2 \notin \mathbb{Z}$, then the parameter κ above is non-zero, provided that the germ f_1 constructed in Definition 2.1 as the commutator of the holonomy maps along the standard geometric generators has non-zero quadratic part. Moreover, if $\kappa \neq 0$, we can further normalize the above equation in such a way that $\kappa = 1$. By one of our genericity hypotheses, f_1 has a non-trivial quadratic part and moreover this property also holds for any foliation whose holonomy group is analytically conjugate to that of $\mathscr{F}$. Therefore, all foliations considered in this work may be normalized in such a way that $\kappa = 1$. We arrive to the following normal form:

$$\frac{dz}{dw} = z\,\frac{s(w)(1+\alpha_0 z) + z + \eta z^2}{r(w)(1+\alpha_0\sigma z) + p(w) z^2}. \tag{4.1}$$

In this way, any generic foliation $\mathscr{F} \in \mathscr{A}_2$ is uniquely defined by five complex parameters: $\lambda_1, \lambda_2, \alpha_0, \alpha_1, \alpha_2$. We write $\mathscr{F} = \mathscr{F}(\lambda, \alpha)$ to emphasize this fact. In what follows, $\widetilde{\mathscr{F}}$ will denote a foliation from $\mathscr{A}_2$ whose holonomy group at infinity is analytically equivalent to that of $\mathscr{F}$. We deduce from such conjugacy and from the non-solvability of the holonomy groups that $\widetilde{\mathscr{F}}$ has the same characteristic numbers at infinity. Therefore, we may write $\widetilde{\mathscr{F}} = \mathscr{F}(\lambda, \beta)$ where $\beta \in \mathbb{C}^3$.

Let us denote the right hand side of (4.1) by $\Psi(z, w)$. The rational function Ψ has a power series expansion with respect to z of the form

$$\Psi(z, w) = \sum_{d=1}^{\infty} K_d(w)\, z^d, \tag{4.2}$$

where K_d is a rational function in w. Since $\Psi(0, w)$ has denominator $r(w)$, we can expect that the rational functions $K_d(w)$ to have $r(w)$ to some power as denominator. We will see in Proposition 4.2 that this is in fact the case and that moreover such power can always be taken to be equal to d.

In particular, the first coefficient $K_1(w)$ is the rational function

$$K_1(w) = \frac{s(w)}{r(w)} = \frac{\lambda_1}{w+1} + \frac{\lambda_2}{w-1}.$$

The first variation of the solution $z = 0$ to Eq. (4.1) satisfies the linear equation

$$\frac{d\varphi_1}{dw} = K_1(w)\,\varphi_1, \quad \varphi_1(0) = 1, \tag{4.3}$$

and so

$$\varphi_1(w) = (1+w)^{\lambda_1}(1-w)^{\lambda_2}, \quad \varphi_1(0) = 1. \tag{4.4}$$

The higher variations φ_d, $d \geq 2$, satisfy an inhomogeneous linear equation whose associate homogeneous equation is (4.3):

$$\frac{d\varphi_d}{dw} = K_1(w)\,\varphi_d + b_d(w), \quad \varphi_d(0) = 0.$$

Let us write $B_d(t) = \varphi_1(t)^{-1}\, b_d(t)$ so that the solution to the above equation is given by

$$\varphi_d(w) = \varphi_1(w) \int_0^w B_d(t)\, dt.$$

Let us define $\phi_d(w) = \int_0^w B_d(t)\,\mathrm{d}t$ and call these functions the *reduced variations*. In this way, $\varphi_d = \varphi_1\phi_d$. The non-linear terms $b_d(w)$ are well known for an equation of the form (2.1). The following proposition gives an explicit expression for $B_d = \varphi_1^{-1}\, b_d$.

Proposition 4.1 *The functions B_d defined above are given by the following formulas:*

$$
\begin{aligned}
B_2 &= K_2\varphi_1,\\
B_3 &= 2K_2\phi_2\varphi_1 + K_3\varphi_1^2,\\
B_4 &= K_2(2\phi_3\varphi_1 + \phi_2^2\varphi_1) + 3K_3\phi_2\varphi_1^2 + K_4\varphi_1^3,\\
B_5 &= 2K_2(\phi_4\varphi_1 + \phi_3\phi_2\varphi_1) + 3K_3(\phi_3\varphi_1^2 + \phi_2^2\varphi_1^2) + 4K_4\phi_2\varphi_1^3 + K_5\varphi_1^4,\\
B_6 &= K_2(2\phi_5\varphi_1 + 2\phi_4\phi_2\varphi_1 + \phi_3^2\varphi_1) + K_3(3\phi_4\varphi_1^2 + 6\phi_3\phi_2\varphi_1^2 + \phi_2^3\varphi_1^2)\\
&\quad + K_4(4\phi_3\varphi_1^3 + 6\phi_2^2\varphi_1^3) + 5K_5\phi_2\varphi_1^4 + K_6\varphi_1^5.
\end{aligned}
$$

To compute the reduced variations $\phi_d(w) = \int_0^w B_d\,\mathrm{d}t$, it will be convenient to split each of the rational functions $K_d(w)$ into two pieces, one of these a scalar multiple of $K_1(w)$. Computations are simplified since, in virtue of (4.3), we can compute explicitly an integral of the form $\int_0^w K_1\varphi_1^m\,\mathrm{d}t$.

Definition 4.1 Given a rational differential equation $\dfrac{\mathrm{d}z}{\mathrm{d}w} = \Psi(z, w)$ normalized as in (4.1), we define the rational function

$$
C(z, w) = z\frac{s(w)(1 + \alpha_0 z)}{r(w)(1 + \alpha_0\sigma z)},
$$

where $s(w), r(w), \sigma$ are as in (4.1). We also define $S(z, w)$ by the formula

$$
\Psi(z, w) = C(z, w) + S(z, w). \tag{4.5}
$$

Remark 4.1 It is proved in [8] that a foliation given by

$$
\frac{\mathrm{d}z}{\mathrm{d}w} = C(z, w),
$$

with $C(z, w)$ as above has a commutative holonomy group. This holonomy group is in fact linearizable, but it is not linear unless $\alpha_0 = 0$.

Note that

$$
C(z, w) = K_1(w)\vartheta(z),
$$

where $\vartheta(z)$ is the rational function $\vartheta(z) = z(1 + \alpha_0 z)(1 + \alpha_0\sigma z)^{-1}$.

Proposition 4.2 *The splitting of $\Psi(z, w)$ given in Eq. (4.5) implies that for each $d \geq 1$,*

$$
K_d(w) = c_d\, K_1(w) + \frac{S_d(w)}{r(w)^d}, \tag{4.6}
$$

where the polynomials $S_d(w)$ *are given by the formula*

$$S(z, w) = \sum_{d=2}^{\infty} \frac{S_d(w)}{r(w)^d} z^d,$$

and the constants c_d *are given by* $\vartheta(z) = \sum_{d=1}^{\infty} c_d z^d$.

Explicit expressions for c_d and S_d in terms of the parameters λ and α are given at the beginning of Sect. 7.

Remark 4.2 We have expanded the distinguished parabolic germs in power series

$$f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \cdots .$$

According to (2.2), we have $a_{dj} = \varphi_{d\{\gamma_j\}}(0)$, and we also know that $\varphi_{1\{\gamma_j\}}(0) = 1$ since the loops γ_1, γ_2 are commutators. The equality $\varphi_d = \varphi_1 \phi_d$ implies that in fact

$$a_{dj} = \phi_{d\{\gamma_j\}}(0).$$

This fact will be used in the next section when computing the coefficients a_{dj}.

5 Analysis of the power series expansion of the distinguished parabolic germs f_j

In this section, we compute the coefficients a_{dj} in the power series expansion of the distinguished parabolic germ f_j. These computations follow very closely computations carried out in [9]. However, in [9] it is assumed that the holonomy group at infinity of the foliation in question is solvable, and thus several simplifications take place. The computations provided here are completely general.

5.1 Analysis of the terms of low degree

Proposition 5.1 *The reduced second variation is given by*

$$\phi_2(w) = c_2(\varphi_1(w) - 1) + \psi_2(w),$$

where

$$\psi_2(w) = \int_0^w \frac{S_2(t)}{r(t)^2} \varphi_1(t)\, \mathrm{d}t,$$

and c_2, S_2 *are as in Proposition 4.2. In particular, we have*

$$a_{2j} = \psi_{2j} \quad \text{with} \quad \psi_{2j} = \int_{\gamma_j} \frac{S_2(w)}{r(w)^2} \varphi_1(w)\, \mathrm{d}w, \quad j = 1, 2.$$

Proof The reduced variation is given by $\phi_2(w) = \int_0^w B_2\,\mathrm{d}t$. It follows from Proposition 4.1 and Eq. (4.6) that

$$\phi_2(w) = \int_0^w K_2(t)\varphi_1(t)\,\mathrm{d}t = \int_0^w \left(c_2 K_1(t) + \frac{S_2(t)}{r(t)^2}\right)\varphi_1(t)\,\mathrm{d}t.$$

Note that

$$\int_0^w K_1(t)\varphi_1(t)\,\mathrm{d}t = \int_0^w \frac{\mathrm{d}\varphi_1}{\mathrm{d}t}\,\mathrm{d}t = \varphi_1(w) - 1,$$

and so

$$\phi_2(w) = c_2(\varphi_1(w) - 1) + \int_0^w \frac{S_2(t)}{r(t)^2}\,\varphi_1(t)\,\mathrm{d}t,$$

as claimed. □

Proposition 5.2 *The reduced third variation is given by*

$$\phi_3(w) = \phi_2(w)^2\varphi_1(w) + c_3\frac{\varphi_1(w)^2 - 1}{2} + \psi_3(w),$$

where

$$\psi_3(w) = \int_0^w \frac{S_3(t)}{r(t)^3}\varphi_1(t)^2\,\mathrm{d}t,$$

and c_3, S_3 are as in Proposition 4.2. In particular,

$$a_{3j} = a_{2j}^2 + \psi_{3j} \quad \textit{with} \quad \psi_{3j} = \int_{\gamma_j} \frac{S_3(w)}{r(w)^3}\varphi_1(w)^2\,\mathrm{d}w, \quad j = 1, 2.$$

Proof By Proposition 4.1, we have that ϕ_3 is given by

$$\int_0^w B_3\,\mathrm{d}t = 2\int_0^w K_2\phi_2\varphi_1\,\mathrm{d}t + \int_0^w K_3\varphi_1^2\,\mathrm{d}t.$$

The first integral on the right hand side can be easily computed:

$$\int_0^w K_2\phi_2\varphi_1\,\mathrm{d}t = \int_0^w B_2\phi_2\,\mathrm{d}t = \int_0^w \frac{\mathrm{d}\phi_2}{\mathrm{d}t}\phi_2\,\mathrm{d}t = \frac{1}{2}\phi_2^2.$$

For the second integral, we split K_3 according to (4.6):

$$\int_0^w K_3\varphi_1^2\,\mathrm{d}t = c_3\int_0^w K_1\varphi_1^2\,\mathrm{d}t + \int_0^w \frac{S_3}{r^3}\varphi_1^2\,\mathrm{d}t = c_3\frac{\varphi_1^2 - 1}{2} + \psi_3.$$

Adding up both integrals gives the desired result. □

5.2 Analysis of the terms of higher degree

For degrees higher than the third, we shall not need an explicit expression for the reduced variation $\phi_d(w)$, so we focus only on the coefficient $a_{dj} = \phi_{d\{\gamma_j\}}(0)$.

We stress that for any given exponent n, we have

$$\int_{\gamma_j} K_1\varphi_1^n \,\mathrm{d}w = 0, \quad j = 1, 2,$$

since $\frac{\mathrm{d}\varphi_1}{\mathrm{d}w} = K_1\varphi_1$ and $\varphi_{1\{\gamma_j\}}(0) = \varphi_1(0) = 1$.

Proposition 5.3 *The coefficient of degree* 4 *in the power series expansion of* f_j *is given by*

$$a_{4j} = 2a_{3j}a_{2j} - a_{2j}^3 + \frac{c_3}{2}a_{2j} - c_2\psi_{3j} + \Delta_{1j} + \psi_{4j},$$

where

$$\Delta_{1j} = \int_{\gamma_j} \frac{S_3(w)}{r(w)^3}\,\psi_2(w)\varphi_1(w)^2 \,\mathrm{d}w, \quad \psi_{4j} = \int_{\gamma_j} \frac{q_4(w)}{r(w)^4}\,\varphi_1(w)^3 \,\mathrm{d}w,$$

and the polynomial $q_4(w)$ *is defined to be*

$$q_4(w) = S_4(w) + c_2S_3(w)r(w) - \frac{c_3}{2}S_2(w)r(w)^2,$$

with the terms c_d, S_d *as in Proposition* 4.2.

Proof By Proposition 4.1, we know that a_{4j} is given by

$$a_{4j} = \int_{\gamma_j} B_4 \,\mathrm{d}w = 2H_{1j} + H_{2j} + 3H_{3j} + H_{4j}, \tag{5.1}$$

where

$$H_{1j} = \int_{\gamma_j} K_2\phi_3\varphi_1 \,\mathrm{d}w, \qquad H_{2j} = \int_{\gamma_j} K_2\phi_2^2\varphi_1 \,\mathrm{d}w,$$

$$H_{3j} = \int_{\gamma_j} K_3\phi_2\varphi_1^2 \,\mathrm{d}w, \qquad H_{4j} = \int_{\gamma_j} K_4\varphi_1^3 \,\mathrm{d}w.$$

We now proceed to compute these integrals. It is straightforward that $H_{2j} = \int_{\gamma_j} B_2\phi_2^2 \,\mathrm{d}w$, hence

$$H_{2j} = \int_{\gamma_j} \left(\frac{1}{3}\phi_2^3\right)' \mathrm{d}w = \frac{1}{3}a_{2j}^3.$$

Note that $H_{1j} = \int_{\gamma_j} B_2\phi_3 \,\mathrm{d}w$, so integration by parts yields

$$\int_{\gamma_j} \frac{\mathrm{d}\phi_2}{\mathrm{d}t}\phi_3 \,\mathrm{d}w = a_{3j}a_{2j} - \int_{\gamma_j} B_3\phi_2 \,\mathrm{d}w.$$

Using the expression for B_3 provided in Proposition 4.1, we see that

$$H_{1j} = a_{3j}a_{2j} - 2\int_{\gamma_j} K_2\varphi_1\phi_2^2 \,\mathrm{d}w - \int_{\gamma_j} K_3\phi_2\varphi_1^2 \,\mathrm{d}w = a_{3j}a_{2j} - \frac{2}{3}a_2^3 - H_{3j}.$$

Equation (5.1) becomes

$$a_{4j} = 2a_{3j}a_{2j} - a_{2j}^3 + H_{3j} + H_{4j}. \tag{5.2}$$

We split K_3 using Eq. (4.6), thus

$$H_{3j} = \int_{\gamma_j} c_3K_1\phi_2\varphi_1^2 \,\mathrm{d}w + \int_{\gamma_j} \frac{S_3}{r^3}\phi_2\varphi_1^2 \,\mathrm{d}w. \tag{5.3}$$

By Proposition 5.1, we have $\phi_2 = c_2(\varphi_1 - 1) + \psi_2$; therefore the first of the integrals above is given by

$$\int_{\gamma_j} c_3K_1(c_2(\varphi_1 - 1) + \psi_2)\varphi_1^2 \,\mathrm{d}w = \int_{\gamma_j} c_3K_1\psi_2\varphi_1^2 \,\mathrm{d}w,$$

since

$$\int_{\gamma_j} K_1(\varphi_1 - 1)\varphi_1^2 \,\mathrm{d}w = 0.$$

Note that integration by parts yields

$$c_3\int_{\gamma_j} K_1\psi_2\varphi_1^2 \,\mathrm{d}w = c_3\int_{\gamma_j} \left(\frac{1}{2}\varphi_1^2\right)'\psi_2 \,\mathrm{d}w = \frac{c_3}{2}a_{2j} - \int_{\gamma_j} \frac{\frac{c_3}{2}S_2}{r^2}\varphi_1^3 \,\mathrm{d}w.$$

On the other hand, the last integral in (5.3) is given by

$$\begin{aligned}\int_{\gamma_j} \frac{S_3}{r^3}(c_2(\varphi_1 - 1) + \psi_2)\varphi_1^2 \,\mathrm{d}w &= \int_{\gamma_j} \frac{c_2S_3}{r^3}\varphi_1^3 \,\mathrm{d}w - c_2\int_{\gamma_j} \frac{S_3}{r^3}\varphi_1^2 \,\mathrm{d}w + \int_{\gamma_j} \frac{S_3}{r^3}\psi_2\varphi_1^2 \,\mathrm{d}w.\\ &= \int_{\gamma_j} \frac{c_2S_3}{r^3}\varphi_1^3 \,\mathrm{d}w - c_2\psi_3 + \Delta_{1j}.\end{aligned}$$

Therefore,

$$H_{3j} = \frac{c_3}{2}a_{2j} - \int_{\gamma_j} \frac{\frac{c_3}{2}S_2}{r^2}\varphi_1^3 \,\mathrm{d}w + \int_{\gamma_j} \frac{c_2S_3}{r^3}\varphi_1^3 \,\mathrm{d}w - c_2\psi_3 + \Delta_{1j}.$$

Lastly, splitting K_4 according to Eq. (4.6), we get

$$H_{4j} = \int_{\gamma_j} \left(c_4 K_1 + \frac{S_4}{r^4} \right) \varphi_1^3 \,\mathrm{d}w = \int_{\gamma_j} \frac{S_4}{r^4} \varphi_1^3 \,\mathrm{d}w.$$

Substituting the above expressions for H_{3j} and H_{4j} in (5.2) and taking into account that we have defined $q_4 = S_4 + c_2 S_3 r - \frac{c_3}{2} S_2 r^2$, we obtain the desired expression for a_{4j}. □

Proposition 5.4 *The coefficient of degree* 5 *in the power series expansion of* f_j *is given by*

$$\begin{aligned} a_{5j} &= 2a_{4j}a_{2j} + \frac{3}{2}a_{3j}^2 - 4a_{3j}a_{2j}^2 + \frac{3}{2}a_{2j}^4 + \frac{c_3}{2}a_{2j}^2 + \frac{2c_4 - c_3 c_2}{3} a_{2j} \\ &\quad + c_2^2 \psi_{3j} - 2c_2 \psi_{4j} - 2c_2 \Delta_{1j} + \Delta_{2j} + 2\Gamma_{1j} + \psi_{5j}, \end{aligned}$$

where

$$\Delta_{2j} = \int_{\gamma_j} \frac{S_3(w)}{r(w)^3}\, \psi_2(w)^2 \varphi_1(w)^2 \,\mathrm{d}w, \qquad \Gamma_{1j} = \int_{\gamma_j} \frac{q_4(w)}{r(w)^4}\, \psi_2(w) \varphi_1(w)^3 \,\mathrm{d}w,$$

$$\psi_{5j} = \int_{\gamma_j} \frac{q_5(w)}{r(w)^5}\, \varphi_1(w)^4 \,\mathrm{d}w,$$

and the polynomial $q_5(w)$ *is defined to be*

$$q_5 = S_5 + 2c_2 S_4 r + c_2^2 S_3 r^2 - \frac{2}{3}(c_4 + c_3 c_2) S_2 r^3$$

with the terms c_d, S_d *as in Proposition* 4.2.

Proof According to Proposition 4.1, we know that a_{5j} is given by

$$\phi_{5\{\gamma_j\}}(0) = \int_{\gamma_j} B_5 \,\mathrm{d}w = 2I_{1j} + 2I_{2j} + 3I_{3j} + 3I_{4j} + 4I_{5j} + I_{6j}, \tag{5.4}$$

where

$$I_{1j} = \int_{\gamma_j} K_2 \phi_4 \varphi_1 \,\mathrm{d}w, \qquad I_{2j} = \int_{\gamma_j} K_2 \phi_3 \phi_2 \varphi_1 \,\mathrm{d}w, \qquad I_{3j} = \int_{\gamma_j} K_3 \phi_3 \varphi_1^2 \,\mathrm{d}w,$$
$$I_{4j} = \int_{\gamma_j} K_3 \phi_2^2 \varphi_1^2 \,\mathrm{d}w, \qquad I_{5j} = \int_{\gamma_j} K_4 \phi_2 \varphi_1^3 \,\mathrm{d}w, \qquad I_{6j} = \int_{\gamma_j} K_5 \varphi_1^4 \,\mathrm{d}w.$$

The first integrals are computed as follows:

$$\begin{aligned}
I_{2j} &= \int_{\gamma_j} \left(\frac{1}{2}\phi_2^2\right)' \phi_3 \,\mathrm{d}w = \frac{1}{2}a_{2j}^2 a_{3j} - \frac{1}{2}\int_{\gamma_j} B_3\phi_2^2 \,\mathrm{d}w \\
&= \frac{1}{2}a_{3j}a_{2j}^2 - \frac{1}{2}\int_{\gamma_j} 2K_2\phi_2^3\varphi_1 \,\mathrm{d}w - \frac{1}{2}\int_{\gamma_j} K_3\phi_2^2\varphi_1^2 \,\mathrm{d}w \\
&= \frac{1}{2}a_{3j}a_{2j}^2 - \frac{1}{4}a_{2j}^4 - \frac{1}{2}I_{4j}.
\end{aligned}$$

$$\begin{aligned}
I_{3j} &= \int_{\gamma_j} (K_3\varphi_1^2)\phi_3 \,\mathrm{d}w = \int_{\gamma_j} (B_3 - 2K_2\phi_2\varphi_1)\phi_3 \,\mathrm{d}w = \frac{1}{2}a_{3j}^2 - 2I_{2j} \\
&= \frac{1}{2}a_{3j}^2 - a_{3j}a_{2j}^2 + \frac{1}{2}a_{2j}^4 + I_{4j}.
\end{aligned}$$

$$\begin{aligned}
I_{1j} &= \int_{\gamma_j} \frac{\mathrm{d}\phi_2}{\mathrm{d}w}\phi_4 \,\mathrm{d}w = a_{4j}a_{2j} - \int_{\gamma_j} B_4\phi_2 \,\mathrm{d}w \\
&= a_{4j}a_{2j} - \int_{\gamma_j} \left(2K_2\phi_3\phi_2\varphi_1 + K_2\phi_2^3\varphi_1 + 3K_3\phi_2^2\varphi_1^2 + K_4\phi_2\varphi_1^3\right) \mathrm{d}w \\
&= a_{4j}a_{2j} - 2I_{2j} - \frac{1}{4}a_{2j}^4 - 3I_{4j} - I_{5j} \\
&= a_{4j}a_{2j} - a_{3j}a_{2j}^2 + \frac{1}{4}a_{2j}^4 - 2I_{4j} - I_{5j}.
\end{aligned}$$

Therefore, Eq. (5.4) becomes

$$a_{5j} = 2a_{4j}a_{2j} + \frac{3}{2}a_{3j}^2 - 4a_{3j}a_{2j}^2 + \frac{3}{2}a_{2j}^4 + I_{4j} + 2I_{5j} + I_{6j}. \tag{5.5}$$

Next, we break K_3 according to (4.6), so $I_{4j} = \int_{\gamma_j}\left(\frac{S_3}{r^3} + c_3K_1\right)\phi_2^2\varphi_1^2 \,\mathrm{d}w$. Now, using Proposition 5.1, we get

$$\begin{aligned}
\int_{\gamma_j} \frac{S_3}{r^3}\phi_2^2\varphi_1^2 \,\mathrm{d}w &= \int_{\gamma_j} \frac{S_3}{r^3}(c_2(\varphi_1 - 1) + \psi_2)^2\varphi_1^2 \,\mathrm{d}w \\
&= \int_{\gamma_j} \frac{S_3}{r^3}(c_2^2(\varphi_1^4 - 2\varphi_1^3 + \varphi_1^2) + 2c_2(\varphi_1^3 - \varphi_2^2)\psi_2 + \psi_2^2\varphi_1^2) \,\mathrm{d}w.
\end{aligned}$$

Let us group under a same integral those terms having the same exponent on φ_1,

$$\int_{\gamma_j} \frac{S_3}{r^3}\phi_2^2\varphi_1^2\,\mathrm{d}w = \int_{\gamma_j} \frac{c_2^2 S_3}{r^3}\varphi_1^4\,\mathrm{d}w + \int_{\gamma_j} \frac{-2c_2^2 S_3 + 2c_2 S_3\psi_2}{r^3}\varphi_1^3\,\mathrm{d}w + \int_{\gamma_j} \frac{c_2^2 S_3 - 2c_2 S_3\psi_2 + S_3\psi_2^2}{r^3}\varphi_1^2\,\mathrm{d}w. \tag{5.6}$$

On the other hand,

$$\int_{\gamma_j} c_3 K_1 \phi_2^2\varphi_1^2\,\mathrm{d}w = c_3\int_{\gamma_j}\left(\frac{1}{2}\varphi_1^2\right)'\phi_2^2\,\mathrm{d}w = \frac{c_3}{2}a_{2j}^2 - c_3\int_{\gamma_j} B_2\phi_2\varphi_1^2\,\mathrm{d}w.$$

The last integral above is given by $\int_{\gamma_j} K_2\phi_2\varphi_1^3\,\mathrm{d}w = \int_{\gamma_j}\left(\frac{S_2}{r^2} + c_2K_1\right)\phi_2\varphi_1^3\,\mathrm{d}w$.

$$\begin{aligned}\int_{\gamma_j} \frac{S_2}{r^2}\phi_2\varphi_1^3\,\mathrm{d}w &= \int_{\gamma_j} \frac{S_2}{r^2}(c_2(\varphi_1 - 1) + \psi_2)\varphi_1^3\,\mathrm{d}w \\ &= \int_{\gamma_j} \frac{c_2 S_2}{r^2}\varphi_1^4\,\mathrm{d}w + \int_{\gamma_j} \frac{-c_2 S_2 + S_2\psi_2}{r^2}\varphi_1^3\,\mathrm{d}w,\end{aligned}$$

and

$$\int_{\gamma_j} c_2 K_1\phi_2\varphi_1^3\,\mathrm{d}w = \int_{\gamma_j} c_2 K_1(c_2(\varphi_1 - 1) + \psi_2)\varphi_1^3\,\mathrm{d}w = c_2\int_{\gamma_j} K_1\psi_2\varphi_1^3\,\mathrm{d}w.$$

The last equality follows from the fact that $\int_{\gamma_j} K_1(\varphi_1^4 - \varphi_1^3)\,\mathrm{d}w = 0$. The integral on the right hand side above can be integrated by parts to obtain

$$\int_{\gamma_j}\left(\frac{1}{3}\varphi_1^3\right)'\psi_2\,\mathrm{d}w = \frac{1}{3}a_{2j} - \frac{1}{3}\int_{\gamma_j}\frac{S_2}{r^2}\varphi_1^4\,\mathrm{d}w.$$

We conclude that

$$\int_{\gamma_j} c_3 K_1\phi_2^2\varphi_1^2\,\mathrm{d}w = \frac{c_3}{2}a_{2j}^2 - \frac{c_3c_2}{3}a_{2j} + \int_{\gamma_j}\frac{c_3c_2S_2 - c_3S_2\psi_2}{r^2}\varphi_1^3\,\mathrm{d}w + \int_{\gamma_j}\frac{-\frac{2}{3}c_3c_2S_2}{r^2}\varphi_1^4\,\mathrm{d}w. \tag{5.7}$$

We proceed in a similar way to compute $2I_{5j} = 2\int_{\gamma_j}\left(\frac{S_4}{r^4} + c_4K_1\right)\phi_2\varphi_1^3\,\mathrm{d}w$. By Proposition 5.1, $\phi_2 = c_2(\varphi_1 - 1) + \psi_2$, so

$$2\int_{\gamma_j}\frac{S_4}{r^4}\phi_2\varphi_1^3\,\mathrm{d}w = 2\int_{\gamma_j}\frac{c_2S_4}{r^4}\varphi_1^4\,\mathrm{d}w + 2\int_{\gamma_j}\frac{-c_2S_4 + S_4\psi_2}{r^4}\varphi_1^3\,\mathrm{d}w. \tag{5.8}$$

On the other hand,

$$2\int_{\gamma_j} c_4 K_1 \phi_2 \varphi_1^3 \,\mathrm{d}w = 2c_4 \int_{\gamma_j} \left(\frac{1}{3}\varphi_1^3\right)' \phi_2 \,\mathrm{d}w = \frac{2c_4}{3} a_{2j} - \frac{2c_4}{3} \int_{\gamma_j} B_2 \varphi_1^3 \,\mathrm{d}w$$
$$= \frac{2c_4}{3} a_{2j} + \int_{\gamma_j} \frac{-\frac{2}{3} c_4 S_2}{r^2} \varphi_1^4 \,\mathrm{d}w, \tag{5.9}$$

since $B_2 = \left(\frac{S_2}{r^2} + c_2 K_1\right) \varphi_1$ and $\int_{\gamma_j} K_1 \varphi_1^4 \,\mathrm{d}w = 0$.

Lastly, note that writing $K_5 = \frac{S_5}{r^5} + c_5 K_1$ immediately yields

$$I_{6j} = \int_{\gamma_j} \frac{S_5}{r^5} \varphi_1^4 \,\mathrm{d}w. \tag{5.10}$$

The formula claimed for a_{5j} is obtained by combining Eqs. (5.5) to (5.10). Indeed, substituting in (5.5) the expressions found in (5.6)–(5.10) yields

$$a_{5j} = 2a_{4j}a_{2j} + \frac{3}{2}a_{3j}^2 - 4a_{3j}a_{2j}^2 + \frac{3}{2}a_{2j}^4 + \frac{c_3}{2}a_{2j}^2$$
$$+ \frac{2c_4 - c_3c_2}{3} a_{2j} + E_{2j} + E_{3j} + E_{4j},$$

where we have grouped all integrals containing φ_1 to the k-th power in a single integral E_{kj} given by the following expressions:

$$E_{2j} = \int_{\gamma_j} \frac{c_2^2 S_3 - 2c_2 S_3 \psi_2 + S_3 \psi_2^2}{r^3} \varphi_1^2 \,\mathrm{d}w = c_2^2 \psi_{3j} - 2c_2 \Delta_{1j} + \Delta_{2j},$$
$$E_{3j} = \int_{\gamma_j} \frac{-2c_2^2 S_3 r + c_3 c_2 S_2 r^2 - 2c_2 S_4 + (2c_2 S_3 r - c_3 S_2 r^2 + 2S_4)\psi_2}{r^4} \varphi_1^3 \,\mathrm{d}w$$
$$= \int_{\gamma_j} \frac{-c_2 q_4 + 2q_4 \psi_2}{r^4} \varphi_1^3 \,\mathrm{d}w = -2c_2 \psi_{4j} + 2\Gamma_{1j},$$
$$E_{4j} = \int_{\gamma_j} \frac{c_2^2 S_3 r^2 - \frac{2}{3} c_3 c_2 S_2 r^3 + 2c_2 S_4 r - \frac{2}{3} c_4 S_2 r^3 + S_5}{r^5} \varphi_1^4 \,\mathrm{d}w$$
$$= \int_{\gamma_j} \frac{q_5}{r^5} \varphi_1^4 \,\mathrm{d}w = \psi_{5j}.$$

This is exactly the expression claimed by Proposition 5.4. □

Proposition 5.5 *The coefficient of degree* 6 *in the power series expansion of* f_j *is given by*

$$\begin{aligned} a_{6j} &= 2a_{5j}a_{2j} + 3a_{4j}a_{3j} - 4a_{4j}a_{2j}^2 - 5a_{3j}^2 a_{2j} + 7a_{3j}a_{2j}^3 - 2a_{2j}^5 \\ &\quad + \frac{c_3}{2}a_{2j}^3 + \left(c_4 - \frac{c_3c_2}{2}\right)a_{2j}^2 + \left(\frac{3c_5}{4} - \frac{c_4c_2}{2} - \frac{c_3^2}{8} + \frac{c_3c_2^2}{4} + \frac{c_3}{2}\psi_{3j}\right)a_{2j} \\ &\quad - \frac{c_2}{2}\psi_{3j}^2 + \left(\frac{c_4}{3} + \frac{c_3c_2}{3} - c_2^3\right)\psi_{3j} + \left(-\frac{c_3}{2} + 3c_2^2\right)\Delta_{1j} - 3c_2\Delta_{2j} + \Delta_{3j} + \Delta_{(1,1)j} \\ &\quad + \left(-\frac{c_3}{2} + 3c_2^2\right)\psi_{4j} - 6c_2\Gamma_{1j} + 3\Gamma_{2j} + \Gamma_{(0,1)j} - 3c_2\psi_{5j} + 3\mathrm{B}_{1j} + \psi_{6j}, \end{aligned}$$

where

$$\begin{aligned} \Delta_{3j} &= \int_{\gamma_j} \frac{S_3}{r^3}\,\psi_2^3\varphi_1^2\,\mathrm{d}w, & \Delta_{(1,1)j} &= \int_{\gamma_j} \frac{S_3}{r^3}\,\psi_2\psi_3\varphi_1^2\,\mathrm{d}w, \\ \Gamma_{2j} &= \int_{\gamma_j} \frac{q_4}{r^4}\,\psi_2^2\varphi_1^3\,\mathrm{d}w, & \Gamma_{(0,1)j} &= \int_{\gamma_j} \frac{q_4}{r^4}\,\psi_3\varphi_1^3\,\mathrm{d}w, \\ \mathrm{B}_{1j} &= \int_{\gamma_j} \frac{q_5}{r^5}\,\psi_2\varphi_1^4\,\mathrm{d}w, & \psi_{6j} &= \int_{\gamma_j} \frac{q_6}{r^6}\,\varphi_1^5\,\mathrm{d}w, \end{aligned}$$

and the polynomial $q_6(w)$ *is defined to be*

$$\begin{aligned} q_6 &= S_6 + 3c_2S_5r + \left(\frac{c_3}{2} + 3c_2^2\right)S_4r^2 + \left(-\frac{c_4}{3} + \frac{c_3c_2}{6} + c_2^3\right)S_3r^3 \\ &\quad + \left(-\frac{3c_5}{4} - \frac{3c_4c_2}{2} - \frac{c_3^2}{8} - \frac{3c_3c_2^2}{4}\right)S_2r^4, \end{aligned}$$

with the terms c_d, S_d *as in Proposition* 4.2.

Proof According to Proposition 4.1, a_{6j} is given by

$$\begin{aligned} \phi_{6\{\gamma_j\}}(0) &= \int_{\gamma_j} B_6\,\mathrm{d}w \\ &= 2J_{1j} + 2J_{2j} + J_{3j} + 3J_{4j} + 6J_{5j} + J_{6j} + 4J_{7j} + 6J_{8j} + 5J_{9j} + J_{10j}, \end{aligned}$$

where

$$J_{1j} = \int_{\gamma_j} K_2\phi_5\varphi_1 \,\mathrm{d}w, \quad J_{2j} = \int_{\gamma_j} K_2\phi_4\phi_2\varphi_1 \,\mathrm{d}w, \quad J_{3j} = \int_{\gamma_j} K_2\phi_3^2\varphi_1 \,\mathrm{d}w,$$
$$J_{4j} = \int_{\gamma_j} K_3\phi_4\varphi_1^2 \,\mathrm{d}w, \quad J_{5j} = \int_{\gamma_j} K_3\phi_3\phi_2\varphi_1^2 \,\mathrm{d}w, \quad J_{6j} = \int_{\gamma_j} K_3\phi_2^3\varphi_1^2 \,\mathrm{d}w,$$
$$J_{7j} = \int_{\gamma_j} K_4\phi_3\varphi_1^3 \,\mathrm{d}w, \quad J_{8j} = \int_{\gamma_j} K_4\phi_2^2\varphi_1^3 \,\mathrm{d}w, \quad J_{9j} = \int_{\gamma_j} K_5\phi_2\varphi_1^4 \,\mathrm{d}w,$$
$$J_{10j} = \int_{\gamma_j} K_6\varphi_1^5 \,\mathrm{d}w.$$

Let us compute some of these integrals. First, define $J_{0j} = \int_{\gamma_j} K_2\phi_3\phi_2^2\varphi_1 \,\mathrm{d}w$. Taking into account the expression for B_4 presented in Proposition 4.1, we have

$$\begin{aligned} J_{2j} &= \int_{\gamma_j} \left(\frac{1}{2}\phi_2^2\right)' \phi_4 \,\mathrm{d}w = \frac{1}{2}a_{4j}a_{2j}^2 - \frac{1}{2}\int_{\gamma_j} B_4\phi_2^2 \,\mathrm{d}w \\ &= \frac{1}{2}a_{4j}a_{2j}^2 - J_{0j} - \frac{1}{10}a_{2j}^5 - \frac{3}{2}J_{6j} - \frac{1}{2}J_{8j}. \end{aligned}$$

Similarly, taking into account the expression found for B_5, we obtain

$$\begin{aligned} J_{1j} &= \int_{\gamma_j} B_2\phi_5 \,\mathrm{d}w = a_{5j}a_{2j} - \int_{\gamma_j} B_5\phi_2 \,\mathrm{d}w \\ &= a_{5j}a_{2j} - 2J_{2j} - 2J_{0j} - 3J_{5j} - 3J_{6j} - 4I_{8j} - J_{9j} \\ &= a_{5j}a_{2j} - a_{4j}a_{2j}^2 + \frac{1}{5}a_{2j}^5 - 3J_{5j} - 3J_{8j} - J_{9j}. \end{aligned}$$

We also have

$$J_{3j} = \int_{\gamma_j} B_2\phi_3^2 \,\mathrm{d}w = a_{3j}^2a_{2j} - 2\int_{\gamma_j} B_3\phi_3\phi_2 \,\mathrm{d}w = a_{3j}^2a_{2j} - 4J_{0j} - 2J_{5j}$$

and

$$\begin{aligned} J_{4j} &= \int_{\gamma_j} (B_3 - 2K_2\phi_2\varphi_1)\phi_4 \,\mathrm{d}w = a_{4j}a_{3j} - \int_{\gamma_j} B_4\phi_3 \,\mathrm{d}w - 2J_{2j} \\ &= a_{4j}a_{3j} - 2J_{3j} - J_{0j} - 3J_{5j} - J_{7j} - 2J_{2j}. \end{aligned}$$

Taking into account the expressions for J_{3j} and J_{2j} above, we obtain

$$J_{4j} = a_{4j}a_{3j} - a_{4j}a_{2j}^2 - 2a_{3j}^2a_{2j} + \frac{1}{5}a_{2j}^5 + 9J_{0j} + J_{5j} + 3J_{6j} - J_{7j} + J_{8j}.$$

We conclude that

$$a_{6j} = 2a_{5j}a_{2j} + 3a_{4j}a_{3j} - 4a_{4j}a_{2j}^2 - 5a_{3j}^2 a_{2j} + \frac{4}{5}a_{2j}^5 + 21J_{0j} + J_{5j} + 7J_{6j} + J_{7j} + 2J_{8j} + 3J_{9j} + J_{10j}.$$

Note that $J_{0j} = \int_{\gamma_j} \left(\frac{1}{3}\phi_2^3\right)' \phi_3 \, dw = \frac{1}{3}a_{3j}a_{2j}^3 - \frac{1}{3}\int_{\gamma_j} B_3\phi_2^3 \, dw$ and $\int_{\gamma_j} B_3\phi_2^3 \, dw = \frac{2}{5}a_{2j}^5 + J_{6j}$. This shows that

$$J_{0j} = \frac{1}{3}a_{3j}a_{2j}^3 - \frac{2}{15}a_{2j}^5 - \frac{1}{3}J_{6j}.$$

We arrive to the following formula for a_{6j},

$$a_{6j} = 2a_{5j}a_{2j} + 3a_{4j}a_{3j} - 4a_{4j}a_{2j}^2 - 5a_{3j}^2 a_{2j} + 7a_{3j}a_{2j}^3 - 2a_{2j}^5 + J_{5j} + J_{7j} + 2J_{8j} + 3J_{9j} + J_{10j}. \tag{5.11}$$

Let us now compute $J_{5j} = \int_{\gamma_j} K_3\phi_3\phi_2\varphi_1^2 \, dw$. We split K_3 according to (4.6) and write $J_{5j} = J_{5j}^{(1)} + J_{5j}^{(2)}$, where

$$J_{5j}^{(1)} = \int_{\gamma_j} \frac{S_3}{r^3}\phi_3\phi_2\varphi_1^2 \, dw, \quad J_{5j}^{(2)} = \int_{\gamma_j} c_3K_1\phi_3\phi_2\varphi_1^2 \, dw.$$

Note first that using Propositions 5.1 and 5.2, we can write $\phi_3\phi_2\varphi_1^2$ as

$$\left(c_2^2(\varphi_1^2 - 2\varphi_1 + 1) + \frac{1}{2}c_3(\varphi_1^2 - 1) + 2c_2(\varphi_1 - 1)\psi_2 + \psi_2^2 + \psi_3\right)(c_2(\varphi_1 - 1) + \psi_2)\,\varphi_1^2;$$

therefore we obtain

$$\begin{aligned}\phi_3\phi_2\varphi_1^2 = {} & c_2^3(\varphi_1^5 - 3\varphi_1^4 + 3\varphi_1^3 - \varphi_1^2) + \frac{1}{2}c_3c_2(\varphi_1^5 - \varphi_1^4 - \varphi_1^3 + \varphi_1^2) \\ & + 3c_2^2(\varphi_1^4 - 2\varphi_1^3 + \varphi_1^2)\psi_2 + \frac{1}{2}c_3(\varphi_1^4 - \varphi_1^2)\psi_2 + 3c_2(\varphi_1^3 - \varphi_1^2)\psi_2^2 \\ & + \psi_2^3\varphi_1^2 + c_2(\varphi_1^3 - \varphi_1^2)\psi_3 + \psi_3\psi_2\varphi_1^2.\end{aligned} \tag{5.12}$$

We substitute the above expression for $\phi_3\phi_2\varphi_1^2$ in $J_{5j}^{(1)}$ and regroup under the same integral those terms having the same power of φ_1 to obtain

$$\begin{aligned}J_{5j}^{(1)} = {} & \int_{\gamma_j} \frac{(c_2^3 + \frac{1}{2}c_3c_2)S_3}{r^3}\varphi_1^5 \, dw \\ & + \int_{\gamma_j} \frac{(-3c_2^3 - \frac{1}{2}c_3c_2)S_3 + (3c_2^2 + \frac{1}{2}c_3)S_3\psi_2}{r^3}\,\varphi_1^4 \, dw\end{aligned}$$

$$+\int_{\gamma_j} \frac{(3c_2^3-\frac{1}{2}c_3c_2)S_3-6c_2^2S_3\psi_2+3c_2S_3\psi_2^2+c_2S_3\psi_3}{r^3}\varphi_1^3\,\mathrm{d}w$$
$$+\int_{\gamma_j} \frac{(-c_2^3+\frac{1}{2}c_3c_2)S_3+(3c_2^2-\frac{1}{2}c_3)S_3\psi_2}{r^3}$$
$$+\int_{\gamma_j} \frac{-3c_2S_3\psi_2^2+S_3\psi_2^3-c_2S_3\psi_3+S_3\psi_3\psi_2}{r^3}\varphi_1^2\,\mathrm{d}w. \tag{5.13}$$

We shall simplify only one of the above terms: Note that

$$\int_{\gamma_j} \frac{-c_2S_3\psi_3}{r^3}\varphi_1^2\,\mathrm{d}w=-c_2\int_{\gamma_j}\frac{d\psi_3}{\mathrm{d}w}\psi_3\,\mathrm{d}w=-\frac{1}{2}c_2\psi_{3j}^2.$$

We thus obtain

$$J_{5j}^{(1)}=-\frac{1}{2}c_2\psi_{3j}^2+\int_{\gamma_j}\frac{(c_2^3+\frac{1}{2}c_3c_2)S_3}{r^3}\varphi_1^5\,\mathrm{d}w$$
$$+\int_{\gamma_j}\frac{(-3c_2^3-\frac{1}{2}c_3c_2)S_3+(3c_2^2+\frac{1}{2}c_3)S_3\psi_2}{r^3}\varphi_1^4\,\mathrm{d}w$$
$$+\int_{\gamma_j} \frac{(3c_2^3-\frac{1}{2}c_3c_2)S_3-6c_2^2S_3\psi_2+3c_2S_3\psi_2^2+c_2S_3\psi_3}{r^3}\varphi_1^3\,\mathrm{d}w$$
$$+\int_{\gamma_j} \frac{(-c_2^3+\frac{1}{2}c_3c_2)S_3+(3c_2^2-\frac{1}{2}c_3)S_3\psi_2}{r^3}$$
$$+\int_{\gamma_j} \frac{-3c_2S_3\psi_2^2+S_3\psi_2^3-c_2S_3\psi_3+S_3\psi_3\psi_2}{r^3}\varphi_1^2\,\mathrm{d}w. \tag{5.14}$$

For computing $J_{5j}^{(2)}=\int_{\gamma_j}c_3K_1\phi_3\phi_2\varphi_1^2\,\mathrm{d}w$, we also substitute the expression for $\phi_3\phi_2\varphi_1^2$ found in (5.12). Splitting the integral into individual terms, we get expressions of the form $\int_{\gamma_j}K_1\psi_3^s\psi_2^t\varphi_1^k\,\mathrm{d}w$. For each of these terms, we use one of the following integration by parts formulas:

$$\int_{\gamma_j}K_1\psi_2^s\varphi_1^k\,\mathrm{d}w=\frac{1}{k}a_{2j}^s-\frac{s}{k}\int_{\gamma_j}\frac{S_2}{r^2}\psi_2^{s-1}\varphi_1^{k+1}\,\mathrm{d}w,$$
$$\int_{\gamma_j}K_1\psi_3^s\varphi_1^k\,\mathrm{d}w=\frac{1}{k}\psi_{3j}^s-\frac{s}{k}\int_{\gamma_j}\frac{S_3}{r^3}\psi_3^{s-1}\varphi_1^{k+2}\,\mathrm{d}w,$$
$$\int_{\gamma_j}K_1\psi_3\psi_2\varphi_1^2\,\mathrm{d}w=\frac{1}{2}\psi_{3j}a_{2j}-\frac{1}{2}\int_{\gamma_j}\frac{S_3}{r^3}\psi_2\varphi_1^4\,\mathrm{d}w-\frac{1}{2}\int_{\gamma_j}\frac{S_2}{r^2}\psi_3\varphi_1^3\,\mathrm{d}w, \tag{5.15}$$

or the fact that $\int_{\gamma_j} K_1\varphi_1^k \,\mathrm{d}w = 0$. After regrouping, we obtain an expression

$$\begin{aligned} J_{5j}^{(2)} &= \frac{1}{2}c_3a_{2j}^3 - \frac{1}{2}c_3c_2a_{2j}^2 + (-\frac{1}{8}c_3^2 + \frac{1}{4}c_3c_2^2)a_{2j} + \frac{1}{2}c_3\psi_{3j}a_{2j} - \frac{1}{6}c_3c_2\psi_{3j} \\ &\quad + \int_{\gamma_j} \frac{(-\frac{1}{8}c_3^2 - \frac{3}{4}c_3c_2^2)S_2r - \frac{1}{3}c_3c_2S_3}{r^3}\,\varphi_1^5\,\mathrm{d}w, \\ &\quad + \int_{\gamma_j} \frac{2c_3c_2^2S_2r + \frac{1}{2}c_3c_2S_3 - 2c_3c_2S_2\psi_2r - \frac{1}{2}c_3S_3\psi_2}{r^3}\,\varphi_1^4\,\mathrm{d}w, \\ &\quad + \int_{\gamma_j} \frac{(\frac{1}{4}c_3^2 - \frac{3}{2}c_3c_2^2)S_2 + 3c_3c_2S_2\psi_2 - \frac{3}{2}c_3S_2\psi_2^2 - \frac{1}{2}c_3S_2\psi_3}{r^2}\,\varphi_1^3\,\mathrm{d}w. \end{aligned} \tag{5.16}$$

Note that by Proposition 5.2, $J_{7j} = \int_{\gamma_j} K_4(\phi_2^2 + \frac{1}{2}c_3(\varphi_1^2 - 1) + \psi_3)\varphi_1^3\,\mathrm{d}w$. Regrouping, we get $J_{7j} = \int_{\gamma_j} K_4(\phi_2^2\varphi_1^3 - \frac{1}{2}c_3\varphi_1^3 + \psi_3\varphi_1^3 + \frac{1}{2}c_3\varphi_1^5)\,\mathrm{d}w$. Since the integral J_{8j} is defined to be $\int_{\gamma_j} K_4\phi_2^2\varphi_1^3\,\mathrm{d}w$, we see that

$$J_{7j} = J_{8j} + \int_{\gamma_j} \left(\frac{S_4}{r^4} + c_4K_1\right)\left(-\frac{1}{2}c_3\varphi_1^3 + \psi_3\varphi_1^3 + \frac{1}{2}c_3\varphi_1^5\right)\mathrm{d}w.$$

Expanding the above product and using the integration by parts formula (5.15), we obtain

$$J_{7j} = J_{8j} + \frac{1}{3}c_4\psi_{3j} + \int_{\gamma_j} \frac{\frac{1}{2}c_3S_4 - \frac{1}{3}c_4S_3r}{r^4}\,\varphi_1^5\,\mathrm{d}w + \int_{\gamma_j} \frac{-\frac{1}{2}c_3S_4 + S_4\psi_3}{r^4}\,\varphi_1^3\,\mathrm{d}w. \tag{5.17}$$

Now, let us also split $J_{8j} = \int_{\gamma_j} K_4\phi_2^2\varphi_1^3\,\mathrm{d}w$ as $J_{8j} = J_{8j}^{(1)} + J_{8j}^{(2)}$ with

$$J_{8j}^{(1)} = \int_{\gamma_j} \frac{S_4}{r^4}\phi_2^2\varphi_1^3\,\mathrm{d}w, \quad J_{8j}^{(2)} = \int_{\gamma_j} c_4K_1\phi_2^2\varphi_1^3\,\mathrm{d}w.$$

Expanding and substituting $\phi_2^2 = (c_2(\varphi_1 - 1) + \psi_2)^2$ into the above expressions, we obtain

$$\begin{aligned} J_{8j}^{(1)} &= \int_{\gamma_j} \frac{c_2^2S_4}{r^4}\,\varphi_1^5\,\mathrm{d}w + \int_{\gamma_j} \frac{-2c_2^2S_4 + 2c_2S_4\psi_2}{r^4}\,\varphi_1^4\,\mathrm{d}w \\ &\quad + \int_{\gamma_j} \frac{c_2^2S_4 - 2c_2S_4\psi_2 + S_4\psi_2^2}{r^4}\,\varphi_1^3\,\mathrm{d}w, \end{aligned} \tag{5.18}$$

$$J^{(2)}_{8j} = \frac{1}{3}c_4 a_{2j}^2 - \frac{1}{6}c_4 c_2 a_{2j} + \int_{\gamma_j} \frac{-\frac{1}{2}c_4 c_2 S_2}{r^2}\,\varphi_1^5\,\mathrm{d}w + \int_{\gamma_j} \frac{\frac{2}{3}c_4 c_2 S_2 - \frac{2}{3}c_4 S_2 \psi_2}{r^2}\,\varphi_1^4\,\mathrm{d}w. \tag{5.19}$$

For the last integrals J_{9j} and J_{10j}, we proceed in an analogous way. We obtain

$$J_{9j} = \frac{1}{4}c_5 a_{2j} + \int_{\gamma_j} \frac{c_2 S_5 - \frac{1}{4}c_5 S_2 r^3}{r^5}\,\varphi_1^5\,\mathrm{d}w + \int_{\gamma_j} \frac{-c_2 S_5 + S_5\psi_2}{r^5}\,\varphi_1^4\,\mathrm{d}w, \tag{5.20}$$

$$J_{10j} = \int_{\gamma_j} \frac{S_6}{r^6}\,\varphi_1^5\,\mathrm{d}w. \tag{5.21}$$

If we now substitute in (5.11) the expressions we have found for $J_{5j}, \ldots, J_{10j}$ given by Eqs. (5.14) to (5.21), we obtain

$$\begin{aligned} a_{6j} &= 2a_{5j}a_{2j} + 3a_{4j}a_{3j} - 4a_{4j}a_{2j}^2 - 5a_{3j}^2 a_{2j} + 7a_{3j}a_{2j}^3 - 2a_{2j}^5 \\ &\quad + \frac{c_3}{2}a_{2j}^3 + \left(c_4 - \frac{c_3 c_2}{2}\right)a_{2j}^2 + \left(\frac{3c_5}{4} - \frac{c_4 c_2}{2} - \frac{c_3^2}{8} + \frac{c_3 c_2^2}{4} + \frac{c_3}{2}\psi_{3j}\right)a_{2j} \\ &\quad - \frac{1}{2}c_2\psi_{3j}^2 + \left(\frac{c_4}{3} - \frac{c_3 c_2}{6}\right)\psi_{3j} + D_{2j} + D_{3j} + D_{4j} + D_{5j}, \end{aligned}$$

where we have grouped all integrals containing φ_1^k into a single expression D_{kj}. These expressions are given explicitly below.

$$\begin{aligned} D_{2j} &= \int_{\gamma_j} \frac{(-c_2^3 + \frac{1}{2}c_3 c_2)S_3 + (3c_2^2 - \frac{1}{2}c_3)S_3\psi_2 - 3c_2 S_3\psi_2^2 + S_3\psi_2^3 + S_3\psi_3\psi_2}{r^3}\,\varphi_1^2\,\mathrm{d}w \\ &= \left(-c_2^3 + \frac{1}{2}c_3 c_2\right)\psi_{3j} + \left(3c_2^2 - \frac{1}{2}c_3\right)\Delta_{1j} - 3c_2\Delta_{2j} + \Delta_{3j} + \Delta_{(1,1)j}. \end{aligned}$$

Recall that $q_4 = S_4 + c_2 S_3 r - \frac{1}{2}c_3 S_2 r^2$. We have:

$$\begin{aligned} D_{3j} &= \int_{\gamma_j} \frac{(3c_2^2 - \frac{1}{2}c_3)(S_4 + c_2 S_3 r - \frac{1}{2}c_3 S_2 r^2) - 6c_2(S_4 + c_2 S_3 r - \frac{1}{2}c_3 S_2 r^2)\psi_2}{r^4}\,\varphi_1^3\,\mathrm{d}w \\ &\quad + \int_{\gamma_j} \frac{3(S_4 + c_2 S_3 r - \frac{1}{2}c_3 S_2 r^2)\psi_2^2 + (S_4 + c_2 S_3 r - \frac{1}{2}c_3 S_2 r^2)\psi_3}{r^4}\,\varphi_1^3\,\mathrm{d}w \\ &= \left(3c_2^2 - \frac{1}{2}c_3\right)\psi_{4j} - 6c_2\Gamma_{1j} + 3\Gamma_{2j} + \Gamma_{(0,1)j}. \end{aligned}$$

Recall also that $q_5 = S_5 + 2c_2S_4r + c_2^2S_3r^2 - \frac{2}{3}(c_4 + c_3c_2)S_2r^3$. Thus,

$$\begin{aligned} D_{4j} &= \int_{\gamma_j} \frac{-3c_2\left(S_5 + 2c_2S_4r + c_2^2S_3r^2 - \frac{2}{3}(c_4 + c_3c_2)S_2r^3\right)}{r^5}\,\varphi_1^4\,\mathrm{d}w \\ &\quad + \int_{\gamma_j} \frac{3\left(S_5 + 2c_2S_4r + c_2^2S_3r^2 - \frac{2}{3}(c_4 + c_3c_2)S_2r^3\right)\psi_2}{r^5}\,\varphi_1^4\,\mathrm{d}w \\ &= -3c_2\psi_{5j} + 3\mathrm{B}_{1j}. \end{aligned}$$

Lastly, we obtain

$$\begin{aligned} D_{5j} = \int_{\gamma_j} &\frac{S_6 + 3c_2S_5r + \left(\frac{1}{2}c_3 + 3c_2^2\right)S_4r^2 + \left(-\frac{1}{3}c_4 + \frac{1}{6}c_3c_2 + c_2^3\right)S_3r^3}{r^6}\,\varphi_1^5\,\mathrm{d}w \\ &+ \int_{\gamma_j} \frac{\left(-\frac{3}{4}c_5 - \frac{3}{2}c_4c_2 - \frac{1}{8}c_3^2 - \frac{3}{4}c_3c_2^2\right)S_2r^4}{r^6}\,\varphi_1^5\,\mathrm{d}w, \end{aligned}$$

which is exactly ψ_{6j}, by definition of $q_6(w)$.

In this way, we obtain exactly the expression claimed by Proposition 5.5, hence concluding its proof. □

6 Proof of the key lemma

We now proceed to prove the key lemma. Let us consider now a normalized foliation $\widetilde{\mathscr{F}}$ whose holonomy group at infinity is analytically conjugate to the holonomy group of $\mathscr{F}$. The genericity assumptions imposed on $\mathscr{F}$ and the way we have normalized imply that both foliations have the same characteristic numbers at infinity at the same singular points. Therefore if $\mathscr{F} = \mathscr{F}(\lambda, \alpha)$, we may write $\widetilde{\mathscr{F}} = \mathscr{F}(\lambda, \beta)$. For every object we have defined for foliation $\mathscr{F}$, we define the analogous object for $\widetilde{\mathscr{F}}$ and denote it by the same symbol with a tilde on top. In particular, $\tilde{f}_1$ and $\tilde{f}_2$ denote the corresponding distinguished parabolic germs which are defined as the holonomy maps along the same loops γ_1 and γ_2 from Definition 2.1. By the conjugacy of the holonomy groups, and in virtue of Remark 2.2, there exists a conformal germ $h \in \mathrm{Diff}(\mathbb{C}, 0)$ such that

$$h \circ f_j - \tilde{f}_j \circ h = 0, \quad j = 1, 2. \tag{6.1}$$

We reemphasize that the idea of the key lemma is to show that the above equation imposes certain conditions on the parameter β. We do this by proving the existence of polynomials $P_d(w)$, whose coefficients depend on λ, α and β, with the property that if Eq. (6.1) holds up to jets of order d, then

$$\int_{\gamma_1} \frac{P_d(w)}{r(w)^d}\,\varphi_1(w)^{d-1}\,\mathrm{d}w = 0.$$

We will first compare the terms of degree 2 in Eq. (6.1) and prove that the normal form (4.1) that we have chosen forces the germ h to be parabolic. The key lemma for degree $d = 3$ will be a corollary of this fact. Once we have done this, we will prove the key lemma for higher degrees, one degree at the time, following the strategy explained in Sect. 3.1.

6.1 Comparison of the terms of low degree

We start with an important observation about the normal form (4.1).

Proposition 6.1 *The polynomial $S_2(w)$ defined in Proposition 4.2 by the property $K_2 = c_2 K_1 + \frac{S_2}{r^2}$ is exactly $S_2(w) = r(w)$. In particular, the function*

$$\psi_2(w) = \int_0^w \frac{S_2}{r^2}\varphi_1 \, \mathrm{d}t = \int_0^w \frac{1}{r}\varphi_1 \, \mathrm{d}t$$

depends only on the characteristic numbers λ_1, λ_2 and not on the parameter α, and so we have $\tilde{\psi}_2(w) = \psi_2(w)$.

This proposition is proved by just expanding $F(z, w)$ in a power series and computing the quadratic coefficient K_2. We omit the proof here since we shall give explicit expression for all the terms S_d and c_d at the beginning of Sect. 7.

Proposition 6.2 *If $h \in \mathrm{Diff}(\mathbb{C}, 0)$ conjugates the holonomy groups of $\mathscr{F}$ and $\widetilde{\mathscr{F}}$, then h is necessarily a parabolic germ and its quadratic coefficient $h_2 = \frac{1}{2}h''(0)$ is given by $h_2 = \tilde{c}_2 - c_2$, with $c_2, \tilde{c}_2$ as in Proposition 4.2.*

Proof If the germ h conjugates the holonomy groups, it conjugates the distinguished parabolic germs, which by genericity hypothesis have non-zero quadratic part. By Proposition 5.1, the quadratic coefficient in the power series of f_j is $a_{2j} = \psi_{2j}$, and by Proposition 6.1 $\psi_2(w)$ depends only on the characteristic numbers λ_1, λ_2. This implies that $a_{2j} = \tilde{a}_{2j}$. Any germ that conjugates two parabolic germs with equal non-zero quadratic part must be parabolic itself, hence h is parabolic.

We now prove the second claim. This is the only instance in this paper where we will consider holonomy maps other than the distinguished parabolic germs. Choose any holonomy map Δ_γ that is not parabolic (for example, choose $\gamma = \mu_1$, a standard geometric generator) and consider its power series expansion: $\Delta_\gamma = \varphi_{1\{\gamma\}}(0)\, z + \varphi_{2\{\gamma\}}(0)\, z^2 + O(z^3)$. We also consider the corresponding power series expansion for $\widetilde{\Delta}_\gamma$. Taking into account that $\widetilde{\varphi}_1 = \varphi_1$, an easy computation shows that $h \circ \Delta_\gamma - \widetilde{\Delta}_\gamma \circ h$ has a power series expansion of the form

$$\left(\varphi_{2\{\gamma\}}(0) - \widetilde{\varphi}_{2\{\gamma\}}(0) + h_2 \varphi_{1\{\gamma\}}(0)(\varphi_{1\{\gamma\}}(0) - 1)\right)\, z^2 + O(z^3),$$

which implies that

$$h_2 = \frac{\widetilde{\varphi}_{2\{\gamma\}}(0) - \varphi_{2\{\gamma\}}(0)}{\varphi_{1\{\gamma\}}(0)(\varphi_{1\{\gamma\}}(0) - 1)},$$

since $h \circ \Delta_\gamma - \widetilde{\Delta}_\gamma \circ h \equiv 0$. Now, we use the relation $\varphi_2 = \varphi_1\phi_2$ and Proposition 5.1 to simplify the numerator. Taking into account that $\psi_2(w) = \tilde{\psi}_2(w)$, we get that $h_2 = \tilde{c}_2 - c_2$. □

We remark that the fact that h is forced to be parabolic depends strongly on the fact that both $\mathscr{F}$ and $\widetilde{\mathscr{F}}$ have been normalized as in (4.1). Without this normalization, the above proposition need not hold.

By virtue of the above proposition, we may write

$$h(z) = z + \sum_{d=2}^{\infty} h_d z^d.$$

Proposition 6.3 *Define* $P_3(w) = \widetilde{S}_3(w) - S_3(w)$. *If a germ* $h \in \mathrm{Diff}(\mathbb{C}, 0)$ *conjugates corresponding pairs of distinguished parabolic germs up to* 3*-jets, then*

$$\int_{\gamma_1} \frac{P_3(w)}{r(w)^3}\, \varphi_1(w)^2 \,\mathrm{d}w = 0.$$

Proof It is easy to check that the commutator of any two parabolic germs is of the form $z + O(z^4)$. This implies that the group of three-jets of parabolic germs is commutative, in particular f_j and $\tilde{f}_j$ have the same 3-jet since $h \circ f_j = \tilde{f}_j \circ h$ and all these germs are parabolic. This tells us that $a_{3j} = \tilde{a}_{3j}$ and moreover $\psi_{3j} = \widetilde{\psi}_{3j}$ since, by Proposition 5.2, $a_{3j} = a_{2j}^2 + \psi_{3j}$, and $\tilde{a}_{2j} = a_{2j}$. Recall that we have defined $\psi_{3j} = \int_{\gamma_j} \frac{S_3}{r^3}\varphi_1^2\,\mathrm{d}w$. Hence,

$$0 = \widetilde{\psi}_{31} - \psi_{31} = \int_{\gamma_1} \frac{\widetilde{S}_3 - S_3}{r^3}\,\varphi_1^2 \,\mathrm{d}w = \int_{\gamma_1} \frac{P_3}{r^3}\,\varphi_1^2\,\mathrm{d}w.$$

□

Before moving on to the key lemma for degree four, we will use Lemma 2.2 to introduce a polynomial $R_3(w)$ needed in the next subsection (see Sect. 3.2 for the general description of the polynomials $R_d(w)$).

Proposition 6.4 *If* $\lambda_1, \lambda_2 \notin \frac{1}{2}\mathbb{Z}$, *there exists a polynomial* $R_3(w)$ *such that*

$$\int_0^w \frac{P_3(t)}{r(t)^3}\,\varphi_1(t)^2\,\mathrm{d}t = \frac{R_3(w)}{r(w)^2}\,\varphi_1(w)^2 - R_3(0).$$

Proof The above proposition is exactly Lemma 2.2 with $P(w) = P_3(w)$ and $u_j = 2\lambda_j - 3$. □

6.2 Key lemma for degree four

In Sect. 3.1, we have reduced the proof of the key lemma on degree 4 to the proof of existence of a polynomial $P_4(w)$ and a complex number $\mathscr{C}_4$, such that

$$a_{2j}\,\mathscr{C}_4 + \mathscr{I}_{4j} = 0, \quad j = 1, 2,$$

where $\mathscr{I}_{4j} = \int_{\gamma_j} \frac{P_4}{r^4}\,\varphi_1^3\,\mathrm{d}w$. Thus, to prove the next proposition, we shall prove the existence of a polynomial P_4 and a number $\mathscr{C}_4$ satisfying the above conditions and cite Proposition 3.1.

Proposition 6.5 *Let $P_4(w) = \tilde{q}_4(w) - q_4(w) - S_2(w)R_3(w)$ with $q_4(w)$ as in Proposition 5.3 and $R_3(w)$ as in Proposition 6.4. If a germ $h \in \mathrm{Diff}(\mathbb{C}, 0)$ conjugates corresponding pairs of distinguished parabolic germs up to 4-jets, then*

$$\int_{\gamma_1} \frac{P_4(w)}{r(w)^4}\,\varphi_1(w)^3\,\mathrm{d}w = 0.$$

Moreover, the cubic coefficient in the power series of h is given by

$$h_3 = h_2^2 + \frac{\tilde{c}_3 - c_3}{2} + R_3(0). \tag{6.2}$$

Proof Taking into account that we know $\tilde{a}_{2j} = a_{2j}$ and $\tilde{a}_{3j} = a_{3j}$, a short computation shows that the coefficient of degree 4 in the power series expansion of $h \circ f_j - \tilde{f}_j \circ h$ is given by $(h_3 - h_2^2)a_{2j} - h_2(a_{3j} - a_{2j}^2) - \tilde{a}_{4j} + a_{4j}$. This implies that

$$\tilde{a}_{4j} - a_{4j} = (h_3 - h_2^2)a_{2j} - h_2(a_{3j} - a_{2j}^2), \quad j = 1, 2. \tag{6.3}$$

On the other hand, it follows from Proposition 5.3 that

$$\tilde{a}_{4j} - a_{4j} = \frac{\tilde{c}_3 - c_3}{2}a_{2j} - (\tilde{c}_2 - c_2)\psi_{3j} + \widetilde{\Delta}_{1j} - \Delta_{1j} + \widetilde{\psi}_{4j} - \psi_{4j}.$$

In the above expression we use the fact that $\tilde{a}_{2j} = a_{2j}$, $\tilde{a}_{3j} = a_{3j}$ and also that $\widetilde{\psi}_{3j} = \psi_{3j}$. Now, using the fact that $\widetilde{\psi}_2(w) = \psi_2(w)$, we see that

$$\widetilde{\Delta}_{1j} - \Delta_{1j} = \int_{\gamma_j} \frac{\widetilde{S}_3 - S_3}{r^3}\psi_2\varphi_1^2\,\mathrm{d}w = \int_{\gamma_j} \frac{P_3}{r^3}\psi_2\varphi_1^2\,\mathrm{d}w.$$

Using Proposition 6.4, we can integrate by parts the last integral above to obtain

$$\widetilde{\Delta}_{1j} - \Delta_{1j} = \int_{\gamma_j} \left(\frac{R_3}{r^2}\varphi_1^2\right)' \psi_2\,\mathrm{d}w = R_3(0)a_{2j} - \int_{\gamma_j} \frac{R_3 S_2}{r^4}\varphi_1^3\,\mathrm{d}w. \tag{6.4}$$

Taking into account that we have defined $P_4 = \tilde{q}_4 - q_4 - S_2 R_3$, we see that

$$\tilde{a}_{4j} - a_{4j} = \frac{\tilde{c}_3 - c_3}{2} a_{2j} - (\tilde{c}_2 - c_2)\psi_{3j} + R_3(0) a_{2j} + \int_{\gamma_j} \frac{P_4}{r^4} \varphi_1^3 \, \mathrm{d}w, \quad j = 1, 2. \tag{6.5}$$

We now substitute the right hand side of (6.3) into (6.5) to obtain an expression

$$(h_3 - h_2^2) a_{2j} - h_2(a_{3j} - a_{2j}^2) = \frac{\tilde{c}_3 - c_3}{2} a_{2j} - (\tilde{c}_2 - c_2)\psi_{3j} + R_3(0) a_{2j} + \int_{\gamma_j} \frac{P_4}{r^4} \varphi_1^3 \, \mathrm{d}w.$$

Recall that $h_2 = \tilde{c}_2 - c_2$ by Proposition 6.2, and recall also that $a_{3j} = a_{2j}^2 + \psi_{3j}$ by Proposition 5.2; therefore, $(\tilde{c}_2 - c_2)\psi_{3j} = h_2(a_{3j} - a_{2j}^2)$. The equation above is thus simplified to

$$(h_3 - h_2^2) a_{2j} = \left(\frac{\tilde{c}_3 - c_3}{2} + R_3(0) \right) a_{2j} + \int_{\gamma_j} \frac{P_4}{r^4} \varphi_1^3 \, \mathrm{d}w,$$

which can be rewritten in the form

$$a_{2j} \mathscr{C}_4 + \mathscr{I}_{4j} = 0,$$

where

$$\mathscr{C}_4 = \frac{\tilde{c}_3 - c_3}{2} + R_3(0) + h_2^2 - h_3,$$

and

$$\mathscr{I}_{4j} = \int_{\gamma_j} \frac{P_4}{r^4} \varphi_1^3 \, \mathrm{d}w.$$

By Proposition 3.1, we have

$$\mathscr{I}_{41} = \int_{\gamma_1} \frac{P_4}{r^4} \varphi_1^3 \, \mathrm{d}w = 0, \qquad \mathscr{C}_4 = 0.$$

This proves the key lemma for degree four. Note that $\mathscr{C}_4 = 0$ implies

$$h_3 = h_2^2 + \frac{\tilde{c}_3 - c_3}{2} + R_3(0).$$

□

We conclude this subsection by introducing the polynomial $R_4(w)$.

Proposition 6.6 *If* $\lambda_1, \lambda_2 \notin \frac{1}{3}\mathbb{Z}$, *there exists a polynomial* $R_4(w)$ *such that*

$$\int_0^w \frac{P_4(t)}{r(t)^4}\,\varphi_1(t)^3\,\mathrm{d}t = \frac{R_4(w)}{r(w)^3}\,\varphi_1(w)^3 + R_4(0).$$

Proof Apply Lemma 2.2 with $P(w) = P_4(w)$ and $u_j = 3\lambda_j - 4$. □

6.3 Key lemma for degree five

We proceed in exactly the same way as we did in the previous subsection.

Proposition 6.7 *Let* $P_5(w) = \tilde{q}_5(w) - q_5(w) - 2S_2(w)R_4(w)$ *with the polynomials* $q_5(w)$ *as in Proposition 5.4 and* $R_4(w)$ *as in Proposition 6.6. If a germ* $h \in \mathrm{Diff}(\mathbb{C}, 0)$ *conjugates corresponding pairs of distinguished parabolic germs up to five-jets, then*

$$\int_{\gamma_1} \frac{P_5(w)}{r(w)^5}\,\varphi_1(w)^4\,\mathrm{d}w = 0.$$

Moreover, the coefficient of degree four in the power series expansion of h *is given by*

$$h_4 = \frac{\tilde{c}_4 - c_4}{3} - \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{6} - R_4(0) - \tilde{c}_2R_3(0) + 3h_3h_2 - 2h_2^3 + \frac{c_3}{2}h_2. \quad (6.6)$$

Proof Taking into account that $\tilde{a}_{2j} = a_{2j}$ and $\tilde{a}_{3j} = a_{3j} = a_{2j}^2 + \psi_{3j}$, a straightforward computation shows that the coefficient of degree 5 in the power series expansion of $h \circ f_j - \tilde{f}_j \circ h$ is given by

$$\begin{aligned} &-\tilde{a}_{5j} + a_{5j} - 4h_2(\tilde{a}_{4j} - a_{4j}) - 2h_2a_{4j} + 2h_2a_{2j}^3 + 3(h_3 - h_2^2)a_{2j}^2 \\ &\quad + (2h_4 - 2h_3h_2 + 2h_2\psi_{3j})a_{2j} - 3h_2^2\psi_{3j}. \end{aligned} \quad (6.7)$$

By Proposition 5.3,

$$a_{4j} = 2(a_{2j}^2 + \psi_{3j})a_{2j} - a_{2j}^3 + \frac{c_3}{2}a_{2j} - c_2\psi_{3j} + \Delta_{1j} + \psi_{4j},$$

and Eq. (6.3) implies

$$\tilde{a}_{4j} - a_{4j} = (h_3 - h_2^2)a_{2j} - h_2\psi_{3j}.$$

Using the above identities and equating (6.7) to zero, we obtain

$$\begin{aligned} \tilde{a}_{5j} - a_{5j} = {} & 3(h_3 - h_2^2)a_{2j}^2 + (2h_4 - 4(h_3 - h_2^2)h_2 - 2h_3h_2 - 2h_2\psi_{3j} - c_3h_2)a_{2j} \\ & + (h_2^2 + 2c_2h_2)\psi_{3j} - 2h_2\Delta_{1j} - 2h_2\psi_{4j}. \end{aligned} \quad (6.8)$$

On the other hand, we can use Proposition 5.4 to compute $\tilde{a}_{5j} - a_{5j}$. We use once more the facts $\tilde{a}_{2j} = a_{2j}$, $\tilde{a}_{3j} = a_{3j}$ and $\tilde{\psi}_{3j} = \psi_{3j}$; thus,

$$\tilde{a}_{5j} - a_{5j} = 2(\tilde{a}_{4j} - a_{4j})a_{2j} + \frac{\tilde{c}_3 - c_3}{2}a_{2j}^2 + \left(2\frac{\tilde{c}_4 - c_4}{3} - \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{3}\right)a_{2j} \tag{6.9}$$

$$+ (\tilde{c}_2^2 - c_2^2)\psi_{3j} - 2\tilde{c}_2\tilde{\psi}_{4j} + 2c_2\psi_{4j} - 2\tilde{c}_2\widetilde{\Delta}_{1j} + 2c_2\Delta_{1j} \tag{6.10}$$

$$+ \widetilde{\Delta}_{2j} - \Delta_{2j} + 2\widetilde{\Gamma}_{1j} - 2\Gamma_{1j} + \tilde{\psi}_{5j} - \psi_{5j}. \tag{6.11}$$

First, note that using the expression found for $\tilde{a}_{4j} - a_{4j}$ in (6.3), we can rewrite the right hand side of (6.9) as

$$\left(2(h_3 - h_2^2) + \frac{\tilde{c}_3 - c_3}{2}\right)a_{2j}^2 + \left(-2h_2\psi_{3j} + 2\frac{\tilde{c}_4 - c_4}{3} - \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{3}\right)a_{2j}. \tag{6.12}$$

Now, note that $\widetilde{\Delta}_{2j} - \Delta_{2j} = \int_{\gamma_j} \frac{P_3}{r^3}\psi_2^2\varphi_1^2\,\mathrm{d}w$, and so integration by parts yields

$$\widetilde{\Delta}_{2j} - \Delta_{2j} = R_3(0)a_{2j}^2 - \int_{\gamma_j} \frac{2R_3S_2}{r^4}\psi_2\varphi_1^3\,\mathrm{d}w. \tag{6.13}$$

Recall that $P_4 = \tilde{q}_4 - q_4 - 2S_2R_3$; therefore,

$$\begin{aligned}\widetilde{\Delta}_{2j} - \Delta_{2j} + 2\widetilde{\Gamma}_{1j} - 2\Gamma_{1j} &= R_3(0)a_{2j}^2 - 2\int_{\gamma_j} \frac{S_2R_3}{r^4}\psi_2\varphi_1^3\,\mathrm{d}w + 2\int_{\gamma_j} \frac{\tilde{q}_4 - q_4}{r^4}\psi_2\varphi_1^3\,\mathrm{d}w \\ &= R_3(0)a_{2j}^2 + 2\int_{\gamma_j} \frac{P_4}{r^4}\psi_2\varphi_1^3\,\mathrm{d}w.\end{aligned}$$

Integrating by parts the last integral, we obtain

$$\int_{\gamma_j} \frac{P_4}{r^4}\psi_2\varphi_1^3\,\mathrm{d}w = -R_4(0)a_{2j} - \int_{\gamma_j} \frac{R_4S_2}{r^5}\varphi_1^4\,\mathrm{d}w.$$

We conclude that

$$\widetilde{\Delta}_{2j} - \Delta_{2j} + 2\widetilde{\Gamma}_{1j} - 2\Gamma_{1j} = R_3(0)a_{2j}^2 - 2R_4(0)a_{2j} - \int_{\gamma_j} \frac{2S_2R_4}{r^5}\varphi_1^4\,\mathrm{d}w.$$

Since we defined $P_5 = \tilde{q}_5 - q_5 - 2S_2R_4$ and $\psi_{5j} = \int_{\gamma_j} \frac{q_5}{r^5}\varphi_1^4\,\mathrm{d}w$, we see that expression (6.11) is given by

$$R_3(0)a_{2j}^2 - 2R_4(0)a_{2j} + \int_{\gamma_j} \frac{P_5}{r^5}\varphi_1^4\,\mathrm{d}w. \tag{6.14}$$

Let us now analyze expression (6.10). Note that $\tilde{c}_2^2 - c_2^2 = h_2^2 + 2c_2h_2$, since $h_2 = \tilde{c}_2 - c_2$, therefore the first term in (6.10) can be rewritten as $(h_2^2 + 2c_2h_2)\psi_{3j}$. Next,

$$-2\tilde{c}_2\tilde{\psi}_{4j} + 2c_2\psi_{4j} = -2h_2\psi_{4j} - 2\tilde{c}_2(\tilde{\psi}_{4j} - \psi_{4j}),$$

and

$$-2\tilde{c}_2\widetilde{\Delta}_{1j} + 2c_2\Delta_{1j} = -2h_2\Delta_{1j} - 2\tilde{c}_2(\widetilde{\Delta}_{1j} - \Delta_{1j}).$$

We have seen already that $\widetilde{\Delta}_{1j} - \Delta_{1j} = R_3(0)a_{2j} - \int_{\gamma_j} \frac{S_2R_3}{r^4}\varphi_1^3\,\mathrm{d}w$, so taking into account that $\psi_{4j} = \int_{\gamma_j} \frac{q_4}{r^4}\varphi_1^3\,\mathrm{d}w$ and $P_4 = \tilde{q}_4 - q_4 - S_2R_3$, we get that expression (6.10) is given by

$$\begin{aligned}(h_2^2 + 2c_2h_2)\psi_{3j} - 2h_2\psi_{4j} - 2h_2\Delta_{1j} - 2\tilde{c}_2R_3(0)a_{2j} - 2\tilde{c}_2\int_{\gamma_j}\frac{P_4}{r^4}\varphi_1^3\,\mathrm{d}w,\\ = (h_2^2 + 2c_2h_2)\psi_{3j} - 2h_2\psi_{4j} - 2h_2\Delta_{1j} - 2\tilde{c}_2R_3(0)a_{2j},\end{aligned} \tag{6.15}$$

since, according to Proposition 6.5, $\int_{\gamma_j} \frac{P_4}{r^4}\varphi_1^3\,\mathrm{d}w = 0$. Adding up all three expressions (6.12), (6.14) and (6.15), and taking into account that

$$h_3 - h_2^2 = \frac{\tilde{c}_3 - c_3}{2} + R_3(0),$$

(which also follows from Proposition 6.5), we finally obtain

$$\begin{aligned}\tilde{a}_{5j} - a_{5j} &= 3(h_3 - h_2^2)a_{2j}^2\\ &\quad + \left(-2h_2\psi_{3j} + 2\frac{\tilde{c}_4 - c_4}{3} - \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{3} - 2R_4(0) - 2\tilde{c}_2R_3(0)\right)a_{2j}\\ &\quad + (h_2^2 + 2c_2h_2)\psi_{3j} - 2h_2\Delta_{1j} - 2h_2\psi_{4j} + \int_{\gamma_j}\frac{P_5}{r^5}\varphi_1^4\,\mathrm{d}w.\end{aligned} \tag{6.16}$$

We now equate the right hand sides of (6.8) and (6.16). Note that we can cancel those terms with a_{2j}^2 as well as those where a_{2j} does not appear, with the exception of $\int_{\gamma_j} \frac{P_5}{r^5}\varphi_1^4\,\mathrm{d}w$. We thus obtain an equation

$$a_{2j}\mathscr{C}_5 + \mathscr{I}_{5j} = 0,$$

where

$$\mathscr{C}_5 = 2\frac{\tilde{c}_4 - c_4}{3} - \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{3} - 2R_4(0) - 2\tilde{c}_2R_3(0) + 6h_3h_2 - 4h_2^3 + c_3h_2 - 2h_4,$$

and

$$\mathscr{I}_{5j} = \int_{\gamma_j} \frac{P_5}{r^5}\,\varphi_1^4\,\mathrm{d}w.$$

By Proposition 3.1,

$$\mathscr{I}_{51} = \int_{\gamma_1} \frac{P_5}{r^5}\,\varphi_1^4\,\mathrm{d}w = 0, \qquad \mathscr{C}_5 = 0.$$

This proves the key lemma for degree five. Moreover, it follows from $\mathscr{C}_5 = 0$ that

$$h_4 = \frac{\tilde{c}_4 - c_4}{3} - \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{6} - R_4(0) - \tilde{c}_2R_3(0) + 3h_3h_2 - 2h_2^3 + \frac{c_3}{2}h_2.$$

Proposition 6.7 is now proved. □

We now introduce the polynomial $R_5(w)$.

Proposition 6.8 *If $\lambda_1, \lambda_2 \notin \frac{1}{4}\mathbb{Z}$, there exists a polynomial $R_5(w)$ such that*

$$\int_0^w \frac{P_5(t)}{r(t)^5}\,\varphi_1(t)^4\,\mathrm{d}t = \frac{R_5(w)}{r(w)^4}\,\varphi_1^4 - R_5(0).$$

Proof Apply Lemma 2.2 with $P(w) = P_5(w)$ and $u_j = 4\lambda_j - 5$. □

6.4 Key lemma for degree six

Proposition 6.9 *Let us define*

$$P_6 = \tilde{q}_6 - q_6 + \tilde{q}_4R_3 - \frac{1}{2}S_2R_3^2 - S_3R_4 - 3S_2R_5,$$

with the polynomials q_6 as in Proposition 5.5 and R_5 as in Proposition 6.8. If a germ $h \in \mathrm{Diff}(\mathbb{C}, 0)$ conjugates corresponding pairs of distinguished parabolic germs up to six-jets then

$$\int_{\gamma_1} \frac{P_6(w)}{r(w)^6}\,\varphi_1(w)^5\,\mathrm{d}w = 0.$$

Proof Let us start by using Proposition 5.5 to obtain an expression for $\tilde{a}_{6j} - a_{6j}$. Using that $\tilde{a}_{2j} = a_{2j}$ and $\tilde{a}_{3j} = a_{3j}$, we obtain the following formula for $\tilde{a}_{6j} - a_{6j}$,

$$2(\tilde{a}_{5j} - a_{5j})a_{2j} + 3(\tilde{a}_{4j} - a_{4j})a_{3j} - 4(\tilde{a}_{4j} - a_{4j})a_{2j}^2 \tag{6.17}$$

$$+ \frac{\tilde{c}_3 - c_3}{2}a_{2j}^3 + \left(\tilde{c}_4 - c_4 - \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{2}\right)a_{2j}^2 \tag{6.18}$$

$$+\left(\frac{3\tilde{c}_5 - 3c_5}{4} - \frac{\tilde{c}_4\tilde{c}_2 - c_4c_2}{2} - \frac{\tilde{c}_3^2 - c_3^2}{8} + \frac{\tilde{c}_3\tilde{c}_2^2 - c_3c_2^2}{4} + \frac{\tilde{c}_3 - c_3}{2}\psi_{3j}\right)a_{2j} \tag{6.19}$$

$$- \frac{\tilde{c}_2 - c_2}{2}\psi_{3j}^2 + \left(\frac{\tilde{c}_4 - c_4}{3} + \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{3} - \tilde{c}_2^3 + c_2^3\right)\psi_{3j} \tag{6.20}$$

$$+ \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right)\widetilde{\Delta}_{1j} - \left(-\frac{c_3}{2} + 3c_2^2\right)\Delta_{1j} \tag{6.21}$$

$$- 3\tilde{c}_2\widetilde{\Delta}_{2j} + 3c_2\Delta_{2j} + \widetilde{\Delta}_{3j} - \Delta_{3j} + \widetilde{\Delta}_{(1,1)j} - \Delta_{(1,1)j} \tag{6.22}$$

$$+ \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right)\tilde{\psi}_{4j} - \left(-\frac{c_3}{2} + 3c_2^2\right)\psi_{4j} - 6\tilde{c}_2\widetilde{\Gamma}_{1j} + 6c_2\Gamma_{1j} + 3\widetilde{\Gamma}_{2j} - 3\Gamma_{2j} \tag{6.23}$$

$$+ \widetilde{\Gamma}_{(0,1)j} - \Gamma_{(0,1)j} - 3\tilde{c}_2\tilde{\psi}_{5j} + 3c_2\psi_{5j} + 3\widetilde{\mathrm{B}}_{1j} - 3\mathrm{B}_{1j} + \tilde{\psi}_{6j} - \psi_{6j}. \tag{6.24}$$

We shall now rewrite several of the terms in the above expression for $\tilde{a}_{6j} - a_{6j}$. For (6.17), we can use the expression for $\tilde{a}_{5j} - a_{5j}$ found in (6.8) and that for $\tilde{a}_{4j} - a_{4j}$ from (6.3), and write $a_{3j} = a_{2j}^2 + \psi_{3j}$. We obtain the following expression after these substitutions:

$$\begin{aligned} &5(h_3 - h_2^2)a_{2j}^3 + \left(4h_4 - 12h_3h_2^2 + 8h_2^3 - 2c_3h_2 - 3h_2\psi_{3j}\right)a_{2j}^2 \\ &+ \left(3h_3\psi_{3j} - h_2^2\psi_{3j} + 4c_2h_2\psi_{3j} - 4h_2\Delta_{1j} - 4h_2\psi_{4j}\right)a_{2j} - 3h_2\psi_{3j}^2. \end{aligned} \tag{6.25}$$

Next, Eq. (6.21) can be rewritten as

$$\left(-\frac{\tilde{c}_3 - c_3}{2} + 3(\tilde{c}_2^2 - c_2^2)\right)\Delta_{1j} + \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right)(\widetilde{\Delta}_{1j} - \Delta_{1j}).$$

We have an expression for $\widetilde{\Delta}_{1j} - \Delta_{1j}$ from Eq. (6.4). Using this, (6.21) becomes

$$\begin{aligned} &\left(-\frac{\tilde{c}_3 - c_3}{2} + 3(\tilde{c}_2^2 - c_2^2)\right)\Delta_{1j} + \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right)R_3(0)a_{2j} \\ &- \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right)\int_{\gamma_j}\frac{S_2R_3}{r^4}\varphi_1^3\,\mathrm{d}w. \end{aligned} \tag{6.26}$$

We also have an expression for $\widetilde{\Delta}_{2j} - \Delta_{2j}$ from (6.13), so the first two terms in (6.22) can be rewritten as

$$\begin{aligned} -3\tilde{c}_2\widetilde{\Delta}_{2j} + 3c_2\Delta_{2j} &= -3(\tilde{c}_2 - c_2)\Delta_{2j} - 3\tilde{c}_2(\widetilde{\Delta}_{2j} - \Delta_{2j}) \\ &= -3(\tilde{c}_2 - c_2)\Delta_{2j} - 3\tilde{c}_2 R_3(0)a_{2j}^2 + 6\tilde{c}_2 \int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \psi_{2j}\varphi_1^3\, \mathrm{d}w. \end{aligned} \tag{6.27}$$

In the same way as we deduced the formulas for $\widetilde{\Delta}_{1j} - \Delta_{1j}$ and $\widetilde{\Delta}_{2j} - \Delta_{2j}$, we integrate $\widetilde{\Delta}_{3j} - \Delta_{3j} = \int_{\gamma_j} \frac{P_3}{r^3}\, \psi_2^3\varphi_1^2\, \mathrm{d}w$ by parts to obtain

$$\widetilde{\Delta}_{3j} - \Delta_{3j} = R_3(0)a_{2j}^3 - 3\int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \psi_2^2\varphi_1^3\, \mathrm{d}w. \tag{6.28}$$

We now wish to express $\widetilde{\Delta}_{(1,1)j} - \Delta_{(1,1)j}$ in terms of simpler objects. We proceed as follows. By definition,

$$\widetilde{\Delta}_{(1,1)j} - \Delta_{(1,1)j} = \int_{\gamma_j} \frac{\widetilde{S}_3}{r^3}\, \tilde{\psi}_2\tilde{\psi}_3\varphi_1^2\, \mathrm{d}w - \int_{\gamma_j} \frac{S_3}{r^3}\, \psi_2\psi_3\varphi_1^2\, \mathrm{d}w,$$

which, taking into account that $\tilde{\psi}_2 = \psi_2$, may be rewritten as

$$\int_{\gamma_j} \frac{\widetilde{S}_3 - S_3}{r^3}\, \psi_2\psi_3\varphi_1^2\, \mathrm{d}w + \int_{\gamma_j} \frac{\widetilde{S}_3}{r^3}\, \psi_2(\tilde{\psi}_3 - \psi_3)\varphi_1^2\, \mathrm{d}w. \tag{6.29}$$

The first integral in the above equation is given by $\int_{\gamma_j} \frac{P_3}{r^3}\, \psi_2\psi_3\varphi_1^2\, \mathrm{d}w$, and so integration by parts yields

$$R_3(0)a_{2j}\psi_{3j} - \int_{\gamma_j} \frac{S_2 R_3}{r^4} \psi_3\varphi_1^3\, \mathrm{d}w - \int_{\gamma_j} \frac{S_3 R_3}{r^5}\, \psi_2\varphi_1^4\, \mathrm{d}w. \tag{6.30}$$

On the other hand, note that

$$\tilde{\psi}_3(w) - \psi_3(w) = \int_0^w \frac{P_3}{r^3}\, \varphi_1^2\, \mathrm{d}t = \frac{R_3(w)}{r(w)^2}\varphi_1(w)^2 - R_3(0). \tag{6.31}$$

Thus, the second integral in (6.29) can be rewritten as

$$\int_{\gamma_j} \frac{\widetilde{S}_3 R_3}{r^5}\, \psi_2\varphi_1^4\, \mathrm{d}w - R_3(0)\widetilde{\Delta}_{1j},$$

since, by definition, $\widetilde{\Delta}_{1j} = \int_{\gamma_j} \frac{\widetilde{S}_3}{r^3}\, \tilde{\psi}_2\varphi_1^2\, \mathrm{d}w$. In fact, taking into account (6.36) and writing $\widetilde{\Delta}_{1j} = \Delta_{1j} + R_3(0)a_{2j} - \int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \varphi_1^3\, \mathrm{d}w$, we obtain that the second integral in (6.29) is given by

$$\int_{\gamma_j} \frac{\widetilde{S}_3 R_3}{r^5}\, \psi_2 \varphi_1^4 \,\mathrm{d}w - R_3(0)\Delta_{1j} - R_3(0)^2 a_{2j} + R_3(0) \int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \varphi_1^3 \,\mathrm{d}w. \quad (6.32)$$

We claim that the following equality holds:

$$\int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \varphi_1^3 \,\mathrm{d}w = \tilde{\psi}_{4j} - \psi_{4j}. \quad (6.33)$$

Indeed, by definition, $\psi_{4j} = \int_{\gamma_j} \frac{q_4}{r^4}\, \varphi_1^3 \,\mathrm{d}w$ and so

$$\tilde{\psi}_{4j} - \psi_{4j} = \int_{\gamma_j} \frac{\tilde{q}_4 - q_4}{r^4}\, \varphi_1^3 \,\mathrm{d}w = \int_{\gamma_j} \frac{P_4}{r^4}\, \varphi_1^3 \,\mathrm{d}w + \int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \varphi_1^3 \,\mathrm{d}w,$$

since we have defined P_4 to be exactly $P_4 = \tilde{q}_4 - q_4 - S_2 R_3$. But according to Proposition 6.5 $\int_{\gamma_j} \frac{P_4}{r^4}\, \varphi_1^3 \,\mathrm{d}w = 0$. This proves our claim and so we deduce that expression (6.32), which is the second integral in (6.29), equals

$$\int_{\gamma_j} \frac{\widetilde{S}_3 R_3}{r^5}\, \psi_2 \varphi_1^4 \,\mathrm{d}w - R_3(0)\Delta_{1j} - R_3(0)^2 a_{2j} + R_3(0)(\tilde{\psi}_{4j} - \psi_{4j}). \quad (6.34)$$

Combining the last integral from (6.30) and the first one from (6.34) into a single integral, we get

$$\begin{aligned} \int_{\gamma_j} \frac{(\widetilde{S}_3 - S_3) R_3}{r^5}\, \psi_2 \varphi_1^4 \,\mathrm{d}w &= \int_{\gamma_j} \frac{P_3 R_3}{r^5}\, \psi_2 \varphi_1^4 \,\mathrm{d}w \\ &= \frac{1}{2} \int_{\gamma_j} \left(\left(\frac{R_3}{r^2}\, \varphi_1^2 \right)^2 \right)' \psi_2 \,\mathrm{d}w \\ &= \frac{1}{2} R_3(0)^2 a_{2j} - \int_{\gamma_j} \frac{\frac{1}{2} S_2 R_3^2}{r^6}\, \varphi_1^5 \,\mathrm{d}w. \end{aligned} \quad (6.35)$$

Combining (6.30) and (6.34), and taking into account (6.35) we obtain the following final expression:

$$\begin{aligned} \widetilde{\Delta}_{(1,1)j} - \Delta_{(1,1)j} = &\left(-R_3(0)\psi_{3j} - \frac{1}{2} R_3(0)^2 \right) a_{2j} - R_3(0)\Delta_{1j} + R_3(0)\tilde{\psi}_{4j} - R_3(0)\psi_{4j} \\ &- \int_{\gamma_j} \frac{S_2 R_3}{r^4} \psi_3 \varphi_1^3 \,\mathrm{d}w - \int_{\gamma_j} \frac{\frac{1}{2} S_2 R_3^2}{r^6}\, \varphi_1^5 \,\mathrm{d}w. \end{aligned} \quad (6.36)$$

Next, we rewrite the first two terms of (6.23) as

$$\left(-\frac{\tilde{c}_3 - c_3}{2} + 3(\tilde{c}_2^2 - c_2^2)\right)\psi_{4j} + \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right)(\tilde{\psi}_{4j} - \psi_{4j}),$$

and use Eq. (6.33) to obtain

$$\left(-\frac{\tilde{c}_3 - c_3}{2} + 3(\tilde{c}_2^2 - c_2^2)\right)\psi_{4j} + \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right)\int_{\gamma_j} \frac{S_2 R_3}{r^4}\,\varphi_1^3\,\mathrm{d}w. \tag{6.37}$$

Similarly,

$$-6\tilde{c}_2\widetilde{\Gamma}_{1j} + 6c_2\Gamma_{1j} = -6(\tilde{c}_2 - c_2)\Gamma_{1j} - 6\tilde{c}_2(\widetilde{\Gamma}_{1j} - \Gamma_{1j}).$$

This time, we claim

$$\widetilde{\Gamma}_{1j} - \Gamma_{1j} = -R_4(0)a_{2j} - \int_{\gamma_j} \frac{S_2 R_4}{r^5}\,\varphi_1^4\,\mathrm{d}w + \int_{\gamma_j} \frac{S_2 R_3}{r^4}\,\psi_2\varphi_1^3\,\mathrm{d}w. \tag{6.38}$$

Indeed, since $P_4 = \tilde{q}_4 - q_4 - S_2 R_3$ and $\Gamma_{1j} = \int_{\gamma_j} \frac{q_4}{r^4}\,\psi_2\varphi_1^3\,\mathrm{d}w$, we have

$$\widetilde{\Gamma}_{1j} - \Gamma_{1j} = \int_{\gamma_j} \frac{P_4}{r^4}\,\psi_2\varphi_1^3\,\mathrm{d}w + \int_{\gamma_j} \frac{S_2 R_3}{r^4}\,\psi_2\varphi_1^3\,\mathrm{d}w.$$

The claimed that formula is simply obtained by integrating by parts the first integral on the right hand side of the above equation. We conclude that

$$\begin{aligned} -6\tilde{c}_2\widetilde{\Gamma}_{1j} + 6c_2\Gamma_{1j} &= -6(\tilde{c}_2 - c_2)\Gamma_{1j} + 6\tilde{c}_2 R_4(0)a_{2j} \\ &\quad + 6\tilde{c}_2\int_{\gamma_j} \frac{S_2 R_4}{r^5}\,\varphi_1^4\,\mathrm{d}w - 6\tilde{c}_2\int_{\gamma_j} \frac{S_2 R_3}{r^4}\,\psi_2\varphi_1^3\,\mathrm{d}w. \end{aligned} \tag{6.39}$$

The analysis for $3\widetilde{\Gamma}_{2j} - 3\Gamma_{2j}$ is analogous:

$$3(\widetilde{\Gamma}_{2j} - \Gamma_{2j}) = 3\int_{\gamma_j} \frac{P_4}{r^4}\,\psi_2^2\varphi_1^3\,\mathrm{d}w + 3\int_{\gamma_j} \frac{S_2 R_3}{r^4}\,\psi_2^2\varphi_1^3\,\mathrm{d}w,$$

which, after integration by parts of the first integral, becomes

$$3\widetilde{\Gamma}_{2j} - 3\Gamma_{2j} = -3R_4(0)a_{2j}^2 - 6\int_{\gamma_j} \frac{S_2 R_4}{r^5}\,\psi_2\varphi_1^4\,\mathrm{d}w + 3\int_{\gamma_j} \frac{S_2 R_3}{r^4}\,\psi_2^2\varphi_1^3\,\mathrm{d}w. \tag{6.40}$$

We now focus on $\widetilde{\Gamma}_{(0,1)j} - \Gamma_{(0,1)j}$. Let us rewrite this expression:

$$\begin{aligned}\widetilde{\Gamma}_{(0,1)j} - \Gamma_{(0,1)j} &= \int_{\gamma_j} \frac{\tilde{q}_4}{r^4}\, \tilde{\psi}_3 \varphi_1^3 \, \mathrm{d}w - \int_{\gamma_j} \frac{q_4}{r^4}\, \psi_3 \varphi_1^3 \, \mathrm{d}w \\ &= \int_{\gamma_j} \frac{\tilde{q}_4 - q_4}{r^4}\, \psi_3 \varphi_1^3 \, \mathrm{d}w + \int_{\gamma_j} \frac{\tilde{q}_4}{r^4}\, (\tilde{\psi}_3 - \psi_3) \varphi_1^3 \, \mathrm{d}w.\end{aligned}$$

The first integral in the last expression above is equal to

$$\int_{\gamma_j} \frac{P_4}{r^4}\, \psi_3 \varphi_1^3 \, \mathrm{d}w + \int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \psi_3 \varphi_1^3 \, \mathrm{d}w,$$

and integrating by parts the first term gives us

$$-R_4(0)\psi_{3j} - \int_{\gamma_j} \frac{S_3 R_4}{r^6}\, \varphi_1^5 \, \mathrm{d}w + \int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \psi_3 \varphi_1^3 \, \mathrm{d}w.$$

Taking into account formula (6.31), we get

$$\int_{\gamma_j} \frac{\tilde{q}_4}{r^4}\, (\tilde{\psi}_3 - \psi_3) \varphi_1^3 \, \mathrm{d}w = \int_{\gamma_j} \frac{\tilde{q}_4 R_3}{r^6}\, \varphi_1^5 \, \mathrm{d}w - R_3(0)\tilde{\psi}_{4j}.$$

We conclude that

$$\begin{aligned}\widetilde{\Gamma}_{(0,1)j} - \Gamma_{(0,1)j} = -R_4(0)\psi_{3j} &- \int_{\gamma_j} \frac{S_3 R_4}{r^6}\, \varphi_1^5 \, \mathrm{d}w + \int_{\gamma_j} \frac{S_2 R_3}{r^4}\, \psi_3 \varphi_1^3 \, \mathrm{d}w \\ &+ \int_{\gamma_j} \frac{\tilde{q}_4 R_3}{r^6}\, \varphi_1^5 \, \mathrm{d}w - R_3(0)\tilde{\psi}_{4j}. \qquad (6.41)\end{aligned}$$

The last terms in (6.24) are as follows: first,

$$\begin{aligned}-3\tilde{c}_2\tilde{\psi}_{5j} + 3c_2\psi_{5j} &= -3(\tilde{c}_2 - c_2)\psi_{5j} - 3\tilde{c}_2(\tilde{\psi}_{5j} - \psi_{5j}) \\ &= -3(\tilde{c}_2 - c_2)\psi_{5j} - 3\tilde{c}_2 \int_{\gamma_j} \frac{\tilde{q}_5 - q_5}{r^5}\, \varphi_1^4 \, \mathrm{d}w \\ &= -3(\tilde{c}_2 - c_2)\psi_{5j} - 6\tilde{c}_2 \int_{\gamma_j} \frac{S_2 R_4}{r^5}\, \varphi_1^4 \, \mathrm{d}w, \qquad (6.42)\end{aligned}$$

since $P_5 = \tilde{q}_5 - q_5 - 2S_2R_4$ and $\int_{\gamma_j} \frac{P_5}{r^5}\, \varphi_1^4 \, \mathrm{d}w = 0$. Second,

$$\begin{aligned}3\widetilde{\mathrm{B}}_{1j} - 3\mathrm{B}_{1j} &= 3\int_{\gamma_j} \frac{P_5}{r^5}\, \psi_2 \varphi_1^4 \, \mathrm{d}w + 6\int_{\gamma_j} \frac{S_2 R_4}{r^5}\, \psi_2 \varphi_1^4 \, \mathrm{d}w \\ &= 3R_5(0)a_{2j} - 3\int_{\gamma_j} \frac{S_2 R_5}{r^6}\, \varphi_1^5 \, \mathrm{d}w + 6\int_{\gamma_j} \frac{S_2 R_4}{r^5}\, \psi_2 \varphi_1^4 \, \mathrm{d}w, \qquad (6.43)\end{aligned}$$

by a simple integration by parts argument.

Under all these modifications, we obtain a new expression for $\tilde{a}_{6j} - a_{6j}$. Moreover, a closer look at the newly found expressions shows that all integrals that appear in such expressions will cancel each other out *except* those in which φ_1 appears raised to the sixth power. Indeed, the integral in (6.26) is canceled out by the integral in (6.37). Similarly, the one in (6.27) and the last integral in (6.39), that in (6.28) and the last integral in (6.40), the first integral in (6.36) and the second one in (6.41), the first integral in (6.39) and the one in (6.42) and the first integral on (6.40) and the last one in (6.43) cancel each other out. We now group all the remaining integrals into a single one. We obtain

$$\int_{\gamma_j} \frac{\tilde{q}_4 R_3 - \frac{1}{2} S_2 R_3^2 - S_3 R_4 - 3 S_2 R_5}{r^6} \varphi_1^5 \, \mathrm{d}w.$$

But recall that we have defined $P_6 = \tilde{q}_6 - q_6 + \tilde{q}_4 R_3 - \frac{1}{2} S_2 R_3^2 - S_3 R_4 - 3 S_2 R_5$ and $\psi_6 = \int_{\gamma_j} \frac{q_6}{r^6} \varphi_1^5 \, \mathrm{d}w$. Since the expression $\tilde{\psi}_{6j} - \psi_{6j}$ appears at the end of (6.24), we can group it with the above integral to obtain a term

$$\int_{\gamma_j} \frac{P_6}{r^6} \varphi_1^5 \, \mathrm{d}w.$$

Note also that the term $R_3(0)\tilde{\psi}_{4j}$ appears in (6.36) and (6.41) with opposite signs, so we cancel out these as well.

We finally obtain a new expression for $\tilde{a}_{6j} - a_{6j}$ from Eqs. (6.25), (6.18), (6.19), (6.20), (6.26), (6.27), (6.28), (6.36), (6.37), (6.39), (6.40), (6.41), (6.42) and (6.43) and taking into account the above considerations.

Formula 1 *The difference $\tilde{a}_{6j} - a_{6j}$ is given by the following expression:*

$$5(h_3 - h_2^2) a_{2j}^3 + \left(4h_4 - 12 h_3 h_2^2 + 8 h_2^3 - 2 c_3 h_2 - 3 h_2 \psi_{3j}\right) a_{2j}^2 \tag{6.44}$$

$$+ \left(3 h_3 \psi_{3j} - h_2^2 \psi_{3j} + 4 c_2 h_2 \psi_{3j} - 4 h_2 \Delta_{1j} - 4 h_2 \psi_{4j}\right) a_{2j} - 3 h_2 \psi_{3j}^2 \tag{6.45}$$

$$+ \frac{\tilde{c}_3 - c_3}{2} a_{2j}^3 + \left(\tilde{c}_4 - c_4 - \frac{\tilde{c}_3 \tilde{c}_2 - c_3 c_2}{2}\right) a_{2j}^2 \tag{6.46}$$

$$+ \left(\frac{3\tilde{c}_5 - 3c_5}{4} - \frac{\tilde{c}_4 \tilde{c}_2 - c_4 c_2}{2} - \frac{\tilde{c}_3^2 - c_3^2}{8} + \frac{\tilde{c}_3 \tilde{c}_2^2 - c_3 c_2^2}{4} + \frac{\tilde{c}_3 - c_3}{2} \psi_{3j}\right) a_{2j} \tag{6.47}$$

$$- \frac{\tilde{c}_2 - c_2}{2} \psi_{3j}^2 + \left(\frac{\tilde{c}_4 - c_4}{3} + \frac{\tilde{c}_3 \tilde{c}_2 - c_3 c_2}{3} - \tilde{c}_2^3 + c_2^3\right) \psi_{3j} \tag{6.48}$$

$$\left(-\frac{\tilde{c}_3 - c_3}{2} + 3(\tilde{c}_2^2 - c_2^2)\right) \Delta_{1j} + \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right) R_3(0) a_{2j} \tag{6.49}$$

$$- 3(\tilde{c}_2 - c_2) \Delta_{2j} - 3 \tilde{c}_2 R_3(0) a_{2j}^2 + R_3(0) a_{2j}^3 \tag{6.50}$$

$$\left(-R_3(0)\psi_{3j} - \frac{1}{2}R_3(0)^2\right) a_{2j} - R_3(0)\Delta_{1j} - R_3(0)\psi_{4j} \tag{6.51}$$

$$\left(-\frac{\tilde{c}_3 - c_3}{2} + 3(\tilde{c}_2^2 - c_2^2)\right) \psi_{4j} - 6(\tilde{c}_2 - c_2)\Gamma_{1j} + 6\tilde{c}_2 R_4(0)a_{2j} \tag{6.52}$$

$$- 3R_4(0)a_{2j}^2 - R_4(0)\psi_{3j} - 3(\tilde{c}_2 - c_2)\psi_{5j} + 3R_5(0)a_{2j} \tag{6.53}$$

$$+ \int_{\gamma_j} \frac{P_6}{r^6}\varphi_1^5 \, \mathrm{d}w. \tag{6.54}$$

We now deduce a second expression for $\tilde{a}_{6j} - a_{6j}$. The coefficient of degree 6 in the power series expansion of $h \circ f_j - \tilde{f}_j \circ h$ is of the form $a_{6j} - \tilde{a}_{6j} + \cdots$. Let us take into account the formulas for a_{3j}, a_{4j} and a_{5j} found in Proposition 5.2, Proposition 5.3 and Proposition 5.4, respectively. Let us also take into account that $\tilde{a}_{2j} = a_{2j}$, $\tilde{a}_{3j} = a_{3j}$ and let us substitute $\tilde{a}_{4j}$ and $\tilde{a}_{5j}$ by their formulas implied by Eqs. (6.3) and (6.8), respectively. Under these considerations, the explicit expression for the coefficient of degree six in $h \circ f_j - \tilde{f}_j \circ h$ may be easily obtained by a simple computed assisted computation.

Formula 2 *The difference $\tilde{a}_{6j} - a_{6j}$ is also given by the following expression:*

$$- 3h_2\psi_{5j} + (-h_3 + 4h_2^2 + 6c_2h_2)\psi_{4j} - \frac{7}{2}h_2\psi_{3j}^2 \tag{6.55}$$

$$+ (h_4 - 2h_3h_2 + c_2h_3 - 4c_2h_2^2 - 3c_2^2h_2)\psi_{3j} \tag{6.56}$$

$$- 6h_2\Gamma_{1j} - 3h_2\Delta_{2j} + (-h_3 + 4h_2^2 + 6c_2h_2)\Delta_{1j} \tag{6.57}$$

$$+ \left(-4h_2\psi_{4j} + (4h_3 - 2h_2^2 + 4c_2h_2)\psi_{3j} - 4h_2\Delta_{1j}\right) a_{2j} \tag{6.58}$$

$$+ \left(3h_5 - 12h_4h_2 - 5h_3^2 - \frac{1}{2}c_3h_3 + 28h_3h_2^2 \right.$$
$$\left. -14h_2^4 + 2c_3h_2^2 - 2c_4h_2 + c_3c_2h_2\right) a_{2j} \tag{6.59}$$

$$+ \left(-3h_2\psi_{3j} + 7h_4 - 21h_3h_2 + 14h_2^3 - \frac{7}{2}c_3h_2\right) a_{2j}^2 \tag{6.60}$$

$$+ 6\left(h_3 - h_2^2\right) a_{2j}^3. \tag{6.61}$$

We now proceed to compare the two formulas above. We shall see once again that everything that depends non-trivially on the index j will be canceled out except for those terms which are a scalar multiple of a_{2j}, and the integral (6.54).

Let us start with those terms having a_{2j}^3. For our first formula, we have such terms on expressions (6.44), (6.46) and (6.50), which add up to

$$\left(5(h_3 - h_2^2) + \frac{\tilde{c}_3 - c_3}{2} + R_3(0)\right) a_{2j}^3.$$

It follows from (6.2) that $h_3-h_2^2=\frac{\tilde{c}_3-c_3}{2}+R_3(0)$, and so the above expression equals $6(h_3-h_2^2)$ which is exactly (6.61), the unique term in Formula 2 having a_{2j}^3.

Consider now those terms with a_{2j}^2. Gathering those in Formula 1 from (6.44), (6.46), (6.50) and (6.53), we get

$$4h_4-12h_3h_2^2+8h_2^3-3h_2\psi_{3j}-2c_3h_2+\tilde{c}_4-c_4-\frac{\tilde{c}_3\tilde{c}_2-c_3c_2}{2}-3\tilde{c}_2R_3(0)-3R_4(0).$$

Using the formula for h_4 from (6.6), we may transform the above expression into

$$\left(7h_4-21h_3h_2+14h_2^3-\frac{7}{2}c_3h_2-3h_2\psi_{3j}\right)a_{2j}^2,$$

which is exactly (6.60).

Let us consider now those terms that have simultaneously a_{2j} *and* something else that depends on the index j. Such terms in Formula 1 appear in (6.45), (6.47) and (6.51). They add up to the following expression:

$$\left(3h_3\psi_{3j}-h_2^2\psi_{3j}+4c_2h_2\psi_{3j}-4h_2\Delta_{1j}-4h_2\psi_{4j}+\frac{\tilde{c}_3-c_3}{2}\psi_{3j}-R_3(0)\psi_{3j}\right)a_{2j}.$$

Substituting $h_3-h_2^2$ instead of $\frac{\tilde{c}_3-c_3}{2}-R_3(0)$, the above turns into

$$\left(4h_3\psi_{3j}-2h_2^2\psi_{3j}+4c_2h_2\psi_{3j}-4h_2\Delta_{1j}-4h_2\psi_{4j}\right)a_{2j},$$

which agrees with (6.58).

Recall that $h_2=\tilde{c}_2-c_2$. Those terms having ψ_{3j}^2 are easily seen to cancel each other out; they are the last term in (6.45) and the first one in (6.48) for Formula 1, and the last term in (6.55) for Formula 2.

Now, let us consider those terms with a single ψ_{3j}. In Formula 1, they appear only in (6.48) and (6.53), and in Formula 2 they are exactly those terms in (6.56). Let us substitute the h_4 term in (6.56) by the expression given in (6.6). Under this substitution, (6.56) becomes

$$\begin{aligned}&\left(\frac{\tilde{c}_4-c_4}{3}-\frac{\tilde{c}_3\tilde{c}_2-c_3c_2}{6}-R_4(0)-\tilde{c}_2R_3(0)\right.\\&\left.+h_3h_2-2h_2^3+\frac{c_3}{2}h_2+c_2h_3-4c_2h_2^2-3c_2^2h_2\right)\psi_{3j}.\end{aligned}\tag{6.62}$$

According to Proposition 6.2 and Eq. (6.2), we have

$$h_2=\tilde{c}_2-c_2 \qquad h_3=\tilde{c}_2^2-2\tilde{c}_2c_2+c_2^2+\frac{\tilde{c}_3-c_3}{2}+R_3(0).$$

Substituting the above expressions into (6.62) yields, after simplification,

$$\left(\frac{\tilde{c}_4 - c_4}{3} + \frac{\tilde{c}_3\tilde{c}_2 - c_3c_2}{3} - R_4(0) - \tilde{c}_2^3 + c_2^3\right)\psi_{3j},$$

which matches exactly those terms in Formula 1 having ψ_{3j}.

The term $(-h_3 + 4h_2^2 + 6c_2h_2)\Delta_{1j}$ in (6.57) may be rewritten, after replacing h_3 by its formula in (6.2), as

$$\left(3h_2^2 - \frac{\tilde{c}_3 - c_3}{2} - R_3(0) + 6c_2h_2\right)\Delta_{1j},$$

which is easily seen to match those terms with Δ_{1j} in (6.49) and (6.51), once we replace h_2 by $\tilde{c}_2 - c_2$.

Note that the terms having ψ_{4j} in (6.55) are $(-h_3+4h_2^2+6c_2h_2)\psi_{4j}$. The coefficient is the same as the coefficient for the Δ_{1j} term we just analyzed, so the same argument shows that this term cancels out those terms in (6.51) and (6.52) having ψ_{4j}.

Taking into account that $h_2 = \tilde{c}_2 - c_2$, it is straightforward that those terms having Δ_{2j}, Γ_{1j} or ψ_{5j} in Formula 1 will cancel out the corresponding ones in Formula 2.

We conclude that equating Formula 1 to Formula 2 yields, after simplification, an equation of the form

$$a_{2j}\,\mathscr{C}_6 + \mathscr{I}_{6j} = 0,$$

where

$$\begin{aligned}\mathscr{C}_6 = {} & \frac{3\tilde{c}_5 - 3c_5}{4} - \frac{\tilde{c}_4\tilde{c}_2 - c_4c_2}{2} - \frac{\tilde{c}_3^2 - c_3^2}{8} + \frac{\tilde{c}_3\tilde{c}_2^2 - c_3c_2^2}{4} \\ & + \left(-\frac{\tilde{c}_3}{2} + 3\tilde{c}_2^2\right)R_3(0) - \frac{1}{2}R_3(0)^2 + 6\tilde{c}_2R_4(0) + 3R_5(0) \\ & - 3h_5 + 12h_4h_2 + 5h_3^2 + \frac{1}{2}c_3h_3 - 28h_3h_2^2 + 14h_2^4 - 2c_3h_2^2 + 2c_4h_2 - c_3c_2h_2,\end{aligned}$$

and

$$\mathscr{I}_{6j} = \int_{\gamma_j} \frac{P_6}{r^6}\,\varphi_1^5\,\mathrm{d}w.$$

By Proposition 3.1,

$$\mathscr{I}_{6j} = \int_{\gamma_j} \frac{P_6}{r^6}\,\varphi_1^5\,\mathrm{d}w = 0, \qquad \mathscr{C}_6 = 0.$$

This proves the key lemma for degree six, and completes the proof of Lemma 2.2. □

Proposition 6.10 *If* $\lambda_1, \lambda_2 \notin \frac{1}{5}\mathbb{Z}$, *there exists a polynomial* $R_6(w)$ *such that*

$$\int_0^w \frac{P_6(t)}{r(t)^6}\,\varphi_1(t)^5\,\mathrm{d}t = \frac{R_6(w)}{r(w)^5}\,\varphi_1(w)^5 + R_6(0).$$

Proof Apply Lemma 2.2 with $P(w) = P_6(w)$ and $u_j = 5\lambda_j - 6$. □

7 Proof of elimination lemma

We have completed the proof of the main lemma, which claims the existence of polynomials F_d, $d = 3, \ldots, 6$, such that if $\mathscr{F}(\lambda, \alpha)$ and $\mathscr{F}(\lambda, \beta)$ have conjugate holonomy groups at infinity, then

$$F_3(\beta) = 0, \ \ldots, \ F_6(\beta) = 0. \tag{7.1}$$

The elimination lemma claims that for generic $(\lambda, \alpha) \in \mathbb{C}^5$, the above polynomial system of equations has a unique solution given by $\beta = \alpha$. To prove such lemma, we need to compute explicit expressions for the polynomials F_d in terms of the parameters α and λ. We can explicitly construct such polynomials F_d following the proof of the key lemma (which is split into Propositions 6.3, 6.5, 6.7, 6.9) and the ideas presented in Sect. 3.2 (deducing the main lemma from the key lemma). All computations in this section were carried out using computer assistance.

Recall that we have defined $F(z, w)$ to be the right hand side of the equation

$$\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{z\,P(z, w)}{Q(z, w)}, \tag{7.2}$$

and that we have defined the rational functions $K_d(w)$ to be the coefficients

$$F(z, w) = \sum_{d=1}^{\infty} K_d(w)\,z^d.$$

We replace $F(z, w)$ by its explicit expression (4.1) and expand it into a power series with respect to z around $z = 0$. After this, we split each coefficient $K_d(w)$ into

$$K_d(w) = c_d\,K_1(w) + \frac{S_d(w)}{r(w)^d},$$

according to Proposition 4.2. We obtain the following expressions for the numbers c_d,

$$c_2 = \alpha_0(1-\sigma), \qquad c_3 = -\alpha_0^2\sigma(1-\sigma), \qquad c_4 = \alpha_0^3\sigma^2(1-\sigma),$$
$$c_5 = -\alpha_0^4\sigma^3(1-\sigma), \qquad c_6 = \alpha_0^5\sigma^4(1-\sigma),$$

and for the polynomials $S_d(w)$,

$$
\begin{aligned}
S_2(w) &= r(w)\\
S_3(w) &= -s(w)p(w)r(w) + (\eta - \alpha_0\sigma)r(w)^2\\
S_4(w) &= -p(w)r(w)^2 + \alpha_0(2\sigma - 1)s(w)p(w)r(w)^2 + \alpha_0\sigma(\alpha_0\sigma - \eta)r(w)^3\\
S_5(w) &= s(w)p(w)^2r(w)^2 + (2\alpha_0\sigma - \eta)p(w)r(w)^3 + \alpha_0^2\sigma(2 - 3\sigma)s(w)p(w)r(w)^3\\
&\quad + \alpha_0^2\sigma^2(\eta - \alpha_0\sigma)r(w)^4\\
S_6(w) &= p(w)^2r(w)^3 + \alpha_0(1 - 3\sigma)s(w)p(w)^2r(w)^3 + (2\alpha_0\sigma\eta - 3\alpha_0^2\sigma^2)p(w)r(w)^4\\
&\quad - \alpha_0^3\sigma^2(3 - 4\sigma)s(w)p(w)r(w)^4 + \alpha_0^3\sigma^3(\alpha_0\sigma - \eta)r(w)^5.
\end{aligned}
$$

Remark 7.1 These computations agree with those presented in [9]. We remark that it is a consequence of the normal form (4.1) we have adopted, that all the above polynomials are divisible by $r(w)$ to some positive power and that $S_2(w)$ does not depend on the parameter α (cf. Proposition 6.1).

7.1 Main lemma revisited

In Sect. 3.2, we have proved the main lemma modulo the auxiliary facts that

$$\deg P_d(w) = 2(d-1), \qquad \text{and} \qquad \deg R_d(w) \le \deg P_d(w) - 1.$$

It follows from a direct inspection of the expressions found for the polynomials $P_d(w)$ in Propositions 6.3, 6.5, 6.7 and 6.9 that for each $d = 3, \ldots, 6$, and the expressions for $S_d(w)$ above, that the polynomial $P_d(w)$ has degree $2(d-1)$. We now show that $\deg R_d(w) \le \deg P_d(w) - 1$ using Lemma 2.2.

Proposition 7.1 *For $d = 3, 4, 5, 6$, the degree of the polynomials $R_d(w)$ satisfy*

$$\deg R_d(w) \le \deg P_d(w) - 1.$$

Proof We know that

$$\int_{\gamma_1} \frac{P_d(w)}{r(w)^d}\varphi_1(w)^{d-1}\,\mathrm{d}w = 0,$$

and we have defined the polynomials $R_d(w)$ by applying Lemma 2.2 with $P(w) = P_d(w)$ and $u_j = (d-1)\lambda_j - d$. Lemma 2.2 also implies that

$$\deg R_d(w) \le \max\left(\deg P_d(w) - 1,\ -2 - \mathrm{Re}(u_1 + u_2)\right).$$

Since $\mathrm{Re}\,\lambda_1 + \mathrm{Re}\,\lambda_2 \ge 2/3$, we conclude that

$$\mathrm{Re}(u_1 + u_2) \ge \frac{2}{3}(d-1) - 2d,$$

and thus

$$-2 - \mathrm{Re}(u_1 + u_2) \leq \frac{4d - 4}{3}.$$

On the other hand, $\deg P_d(w) = 2(d-1)$ and

$$2(d-1) - 1 \geq \frac{4d-4}{3}$$

for any $d \geq 3$. □

7.2 Computing the polynomials F_d

Now that the main lemma has been fully proved, we shall explain how to get explicit expressions for the polynomials $F_d(w)$. In the next subsection, we use these explicit expressions to prove the elimination lemma.

Suppose $P_d(w)$ has degree m, and so $R_d(w)$ has degree at most $m-1$. Let V_m, V_{m-1} denote the vector spaces of polynomials in w of degree at most m and $m-1$, respectively. We have seen in Sect. 3.2, Eq. (3.8), that

$$P_d = R_d' r + (d-1)(s - r')R_d.$$

Consider now the linear map

$$L_d \colon V_{m-1} \longrightarrow V_m, \qquad f(w) \longmapsto f'(w)r(w) + (d-1)(s(w) - r'(w))f(w),$$

where $s(w)$ and $r(w)$ are the polynomials defined in Sect. 4. We prove below that the map L_d has maximal rank and so its image $L_d(V_{m-1})$ is a hyperplane in V_m. Any hyperplane is given by the kernel of some (fixed) linear functional T_d. We have that $\int_{\gamma_1} \frac{P_d}{r^d}\varphi_1^{d-1}\,\mathrm{d}t = 0$ if and only if P_d belongs to the image of L_d, if and only if $T_d(P_d) = 0$. Since the coefficients of P_d are polynomials on β, the expression $T_d(P_d)$ is also a polynomial on β. In this way, we have that $F_d := T_d(P_d)$ vanishes if $\int_{\gamma_1} \frac{P_d}{r^d}\varphi_1^{d-1}\,\mathrm{d}t = 0$.

Proposition 7.2 *The linear map*

$$L_d \colon V_{2d-3} \longrightarrow V_{2d-2}, \qquad f \longmapsto f'r + (d-1)(s - r')f$$

has, with respect to the standard bases $\{1, w, \dots, w^{2d-3}\}$ *and* $\{1, w, \dots, w^{2d-2}\}$*, the following matrix representation*

$$M_d = \begin{pmatrix} A_d & -1 & 0 & \cdots & 0 & 0 & 0 \\ B_d - 2d + 2 & A_d & -2 & \cdots & 0 & 0 & 0 \\ 0 & B_d - 2d + 3 & A_d & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_d - 3 & A_d & -2d + 3 \\ 0 & 0 & 0 & \cdots & 0 & B_d - 2 & A_d \\ 0 & 0 & 0 & \cdots & 0 & 0 & B_d - 1 \end{pmatrix},$$

where

$$A_d = (d-1)(-\lambda_1 + \lambda_2), \qquad B_d = (d-1)(\lambda_1 + \lambda_2).$$

In particular, if $\lambda_3 \notin \frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z}$*, then the linear map* L_d *has maximal rank for each* $d = 3, \dots, 6$.

Proof Obtaining the expression for the above matrix is a straightforward computation. Note that if we drop the first row in the above matrix, we obtain an upper triangular $2(d-1) \times 2(d-1)$ matrix whose diagonal entries are of the form $B_d - k = (d-1)(\lambda_1 + \lambda_2) - k$ with $k = 1, \dots, 2d-2$. Note moreover that such an expression may vanish only if

$$\lambda_3 = 1 - \lambda_1 - \lambda_2 \in \frac{1}{d-1}\mathbb{Z}.$$

This shows that under our genericity assumptions, the matrix M_d, $d = 3, \dots, 6$, has maximal rank. □

Remark 7.2 Let $\tilde{M}_d$ be the $2(d-1) \times 2(d-1)$ matrix obtained by dropping the first row of M_d. Also, let us denote by $\tilde{V}_{2d-2} \subset V_{2d-2}$ the subspace of polynomials without a constant term. If we compose the map L_d with the natural projection $V_{2d-2} \to \tilde{V}_{2d-2}$, we obtain a linear map $\tilde{L}_d \colon V_{2d-3} \to \tilde{V}_{2d-2}$ whose matrix representation is precisely $\tilde{M}_d$. Since $\tilde{M}_d$ is invertible, we conclude that $\tilde{L}_d$ is an isomorphism.

To compute the polynomials R_d and F_d, we input the expressions for c_k, $\tilde{c}_k$, $S_k(w)$, $\widetilde{S}_k(w)$ and $R_k(w)$ for each $k < d$. We compute an explicit expression for the polynomial $P_d(w)$ in terms of λ, α, β according to the formulas found throughout Sect. 6. The polynomial $R_d(w)$ is the unique preimage of $P_d(w)$ under the linear map L_d. We can compute this preimage by inverting the isomorphism $\tilde{L}_d$ defined in Remark 7.2. Indeed, the projection of $P_d(w)$ onto $\tilde{V}_{2d-2}$ is given by $P_d(w) - P_d(0)$ and thus we can find $R_d(w)$ by solving the linear equation

$$\tilde{L}_d(R_d)(w) = P_d(w) - P_d(0) \in \tilde{V}_{2d-2}.$$

Once an expression for $R_d(w)$ has been found, we have that $L_d(R_d)(w)$ and $P_d(w)$ agree on every monomial of positive degree (i.e., they have the same projections onto

$\tilde{V}_{2d-2}$). The condition $L_d(R_d)(w) = P_d(w)$ is thus reduced to the equation

$$L_d(R_d)(0) = P_d(0).$$

The equation $F_d = L_d(R_d)(0) - P_d(0)$ gives us therefore an explicit expression for F_d. Such expressions are quite complicated and so we do not include them here.

7.3 Concluding the elimination lemma

Recall that we have defined the series of resultants

$$\begin{aligned} \mathrm{Res}^1_j(\beta_0,\beta_1) &= \mathrm{Res}_{\beta_2}\left(F_3(\beta_0,\beta_1,\beta_2), F_j(\beta_0,\beta_1,\beta_2)\right), && j=4,5,6, \\ \mathrm{Res}^2_j(\beta_0) &= \mathrm{Res}_{\beta_1}\left(\mathrm{Res}^1_4(\beta_0,\beta_1), \mathrm{Res}^1_j(\beta_0,\beta_1)\right), && j=5,6, \\ \mathrm{Res}^3_6 &= \mathrm{Res}_{\beta_0}\left(\mathrm{Res}^2_5(\beta_0)/(\beta_0-\alpha_0), \mathrm{Res}^2_6(\beta_0)\right), \end{aligned}$$

and proved in Proposition 3.2 that if $\mathrm{Res}^3_6 \not\equiv 0$ as a function of λ and α, then any solution $(\beta_0, \beta_1, \beta_2)$ to system (7.1) satisfies $\beta_0 = \alpha_0$.

After finding explicit expressions for the polynomials F_d, we have computed the above resultants and verified that $\mathrm{Res}^3_6 \not\equiv 0$ zero by evaluating it at the values

$$\lambda_1 = 2 - i, \quad \lambda_2 = 2i, \quad \alpha_0 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = 0, \tag{7.3}$$

and obtaining a non-zero complex number.

The final step in the proof is proving Proposition 3.3. The determinant of the linear system

$$F_3(\alpha_0,\beta_1,\beta_2) = 0, \qquad F_4(\alpha_0,\beta_1,\beta_2) = 0$$

is also obtained with computer assistance and verified to be non-zero at the values of λ and α given in (7.3). All these computations can be found in the appendix of [11]. This completes the proof of the elimination lemma and thus complete also the proof of Theorem 1.1.

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