Ideal Rigidity of Quadratic Foliations on $\mathbb{C}P^2$

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Main Result

In this poster we consider holomorphic foliations on $\mathbb{C}P^2$ form the class $\mathcal{A}_2$; those foliations which in a *fixed* affine neighbourhood $\mathbb{C}^2 \approx \mathbb{C}P^2 \setminus \mathbb{I}$ are generated by a quadratic vector field. The objective of the poster is to present some results about topological rigidity of such foliations which are appear on [6].

Theorem. *A generic foliation from the class $\mathcal{A}_2$ has a neighbourhood in this class such that any other foliation in this neighbourhood which is topologically equivalent to the first foliation must be affine equivalent to the original foliation. No assumptions are being made about the conjugating homeomorphism.*

The key step in the proof of this statement is showing that topologically equivalent foliations must have their monodromy groups at infinity analytically conjugated *in a canonical way.*

Genericity Assumptions

Let us consider foliations $\mathcal{F}$ from the class $\mathcal{A}_2$ which satisfy the following conditions:

(i) The line at infinity $\mathbb{I}$ is invariant;
(ii) The foliation $\mathcal{F}$ only has hyperbolic singularities;
(iii) The monodromy group at infinity $G_{\mathcal{F}}$ is non-solvable.

The first two conditions determinate a (complex and real, respectively) Zariski open set on $\mathcal{A}_2$. The genericity of condition (iii) is discussed in [7].

Key Lemma

Let $\mathcal{F}$ be a generic foliation from $\mathcal{A}_2$. Denote by $\Sigma = \operatorname{Sing}(\mathcal{F}) \cap \mathbb{I} = \{a_1, a_2, a_3\}$. Let D_1, D_2, D_3 be open disks centred at a_1, a_2, a_3 respectively and define $D = \cup D_i$. Let us choose a base point $b \in \mathbb{I} \setminus D$.

Denote by U the set of foliations in $\mathcal{A}_2$ with invariant line $\mathbb{I}$ and having their singularities at infinity on D; one singularity on each D_i.

We can choose canonical generators of $\pi_1(\mathbb{I} \setminus \Sigma, b)$, say γ_1, γ_2 in such a way that for any other foliation $\widetilde{\mathcal{F}}$ the loops γ_1, γ_2 above will still be canonical generators of $\pi_1(\mathbb{I} \setminus \widetilde{\Sigma}, b)$. In this way, for every $\widetilde{\mathcal{F}} \in U$ we obtain canonical generators $\widetilde{\Delta}_{\gamma_1}$, $\widetilde{\Delta}_{\gamma_2}$ of the monodromy group at infinity $G_{\widetilde{\mathcal{F}}}$.

Lemma. *Let $\mathcal{F}, \widetilde{\mathcal{F}}$ be generic foliations from $\mathcal{A}_2$ which are topologically conjugated and sufficiently close. Then their monodromy groups at infinity are canonically conjugated; there exists a germ of biholomorphism $h_0 \colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ that conjugates the canonical generators of the groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$:*

$$h_0 \circ \Delta_{\gamma_i} = \widetilde{\Delta}_{\gamma_i} \circ h_0.$$

Induced Automorphism on the Fundamental Group

Suppose $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are topologically conjugated by a homeomorphism $\mathcal{H}$. We can assume that $\mathcal{H}$ fixes the point b. Both of the infinite leaves $\mathbb{I} \setminus \Sigma$ and $\mathbb{I} \setminus \widetilde{\Sigma}$ deformation retract onto the manifold $\mathbb{I} \setminus D$.

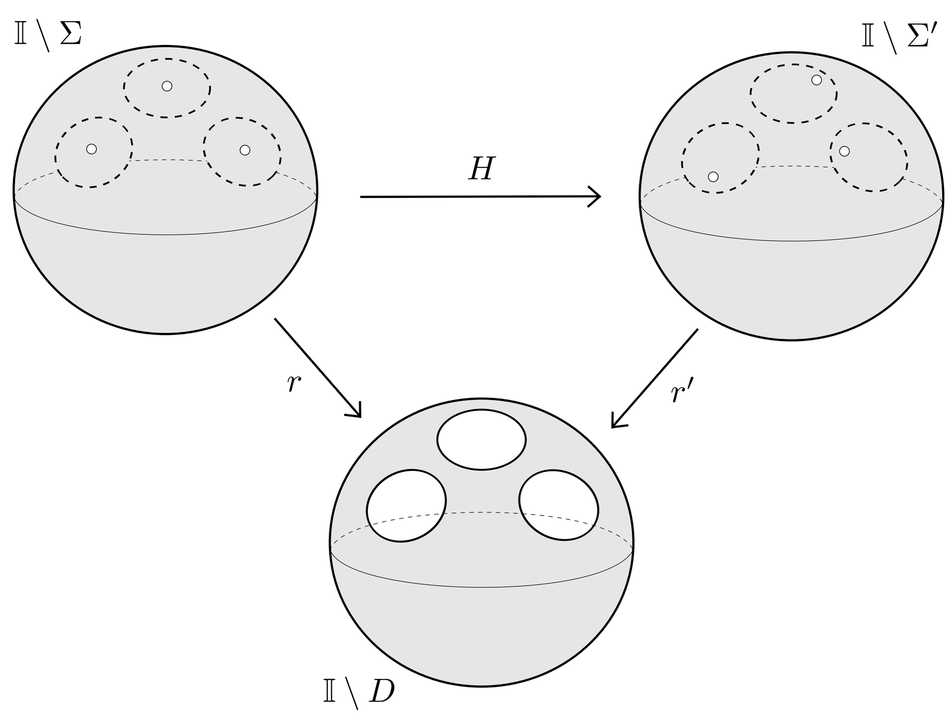

Figure 1

Denote by H the restriction $H = \mathcal{H}|_{\mathbb{I}\setminus\Sigma}$. The homeomorphism H induces an automorphism of the group $\pi_1(\mathbb{I} \setminus D, b)$ in the following way:

$$\begin{array}{ccc} \pi_1(\mathbb{I} \setminus \Sigma, b) & \xrightarrow{H_*} & \pi_1(\mathbb{I} \setminus \widetilde{\Sigma}, b) \\ \downarrow r_* & & \downarrow r'_* \\ \pi_1(\mathbb{I} \setminus D, b) & \xrightarrow[\Phi(H)]{} & \pi_1(\mathbb{I} \setminus D, b) \end{array}$$

We get a well defined map

$$\Phi \colon \operatorname{Topo}(\mathcal{F}) \to \operatorname{Aut}(\pi_1(\mathbb{I} \setminus D, b)).$$

Where $\operatorname{Topo}(\mathcal{F})$ denotes the set of all pairs $(\widetilde{\mathcal{F}}, \mathcal{H})$ in $U \times \operatorname{Homeo}(\mathbb{C}P^2)$ such that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are topologically conjugated by $\mathcal{H}$ and the homeomorphism $\mathcal{H}$ fixes the point b.

Idea Behind the Proof

Suppose $\mathcal{H}$ conjugates foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ and induces the automorphism $\Phi(H) \in \operatorname{Aut}(\pi_1(\mathbb{I} \setminus D, b))$. If $\Phi(H)$ is the identity then the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are canonically conjugated.

Question: Are there any other values of Φ which imply that the monodromy groups are canonically conjugated?

Suppose $\Phi(H)$ is an inner automorphism on $\pi_1(\mathbb{I} \setminus D, b)$. Namely there is an element $\lambda \in \pi_1(\mathbb{I} \setminus D, b)$ such that $\Phi(H)$ is defined by

$$\Phi(H)(\gamma) = \lambda \cdot \gamma \cdot \lambda^{-1}.$$

Then we know there exists an analytic germ $h \colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ that conjugates the monodromy groups at infinity of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ in the following way:

$$h \circ \Delta_{\Phi(H)^{-1}(\gamma)} = \widetilde{\Delta}_{\gamma} \circ h$$

But $\Phi(H)^{-1}(\gamma) = \lambda^{-1} \cdot \gamma \cdot \lambda$ and so

$$\Delta_{\Phi(H)^{-1}(\gamma)} = \Delta_{\lambda} \circ \Delta_{\gamma} \circ \Delta_{\lambda}^{-1}.$$

This implies

$$h \circ \Delta_{\lambda} \circ \Delta_{\gamma} \circ \Delta_{\lambda}^{-1} = \widetilde{\Delta}_{\gamma} \circ h$$

And so

$$h_0 \circ \Delta_{\gamma} = \widetilde{\Delta}_{\gamma} \circ h_0$$

where h_0 is defined to be $h \circ \Delta_{\lambda}$.

Answer: If $\Phi(H)$ is an inner automorphism then the monodromy groups $G_{\mathcal{F}}$ and $G_{\widetilde{\mathcal{F}}}$ are canonically conjugated.

Induced Automorphism on the First Homology Group

In the same way we defined the action Φ on the fundamental group of $\mathbb{I} \setminus D$ we can define a map

$$\begin{array}{rcl} \Psi : \operatorname{Topo}(\mathcal{F}) & \longrightarrow & \operatorname{Aut}(H_1(\mathbb{I} \setminus D; \mathbb{Z})) \\ (\mathcal{F}, \mathcal{H}) & \longmapsto & \Psi(H) \end{array}$$

where $\Psi(H)$ is the automorphism on the first homology group of $\mathbb{I} \setminus D$ defined by

$$\begin{array}{ccc} H_1(\mathbb{I} \setminus \Sigma; \mathbb{Z}) & \xrightarrow{H_*} & H_1(\mathbb{I} \setminus \Sigma'; \mathbb{Z}) \\ \downarrow r_* & & \downarrow r'_* \\ H_1(\mathbb{I} \setminus D; \mathbb{Z}) & \xrightarrow[\phi(H)]{} & H_1(\mathbb{I} \setminus D; \mathbb{Z}) \end{array}$$

Topological Invariance of the Characteristic Numbers

The characteristic numbers of the singular points at infinity are topological invariants for generic foliations. Generically they are pairwise distinct.

Lemma. *There exists a neighbourhood $\Omega \subseteq U$ of $\mathcal{F}$ in $\mathcal{A}_2$ such that for any other foliation $\widetilde{\mathcal{F}} \in \Omega$ topologically conjugated to $\mathcal{F}$ by a homeomorphism $\mathcal{H} \colon \mathbb{C}P^2 \to \mathbb{C}P^2$ the homeomorphism H preserves the numbering of the singular points at infinity; this is, for every $i = 1, 2, 3$ $H(a_i) \in D_i$.*

The Action on Homology is Always Trivial

From the fact that $H \colon \mathbb{I} \setminus \Sigma \to \mathbb{I} \setminus \widetilde{\Sigma}$ is an orientation preserving homeomorphism which extends to all of $\mathbb{I}$ and satisfies the condition $H(a_i) \in D_i$ it can easily be deduced that the automorphism $\Psi(H)$ is always the identity map.

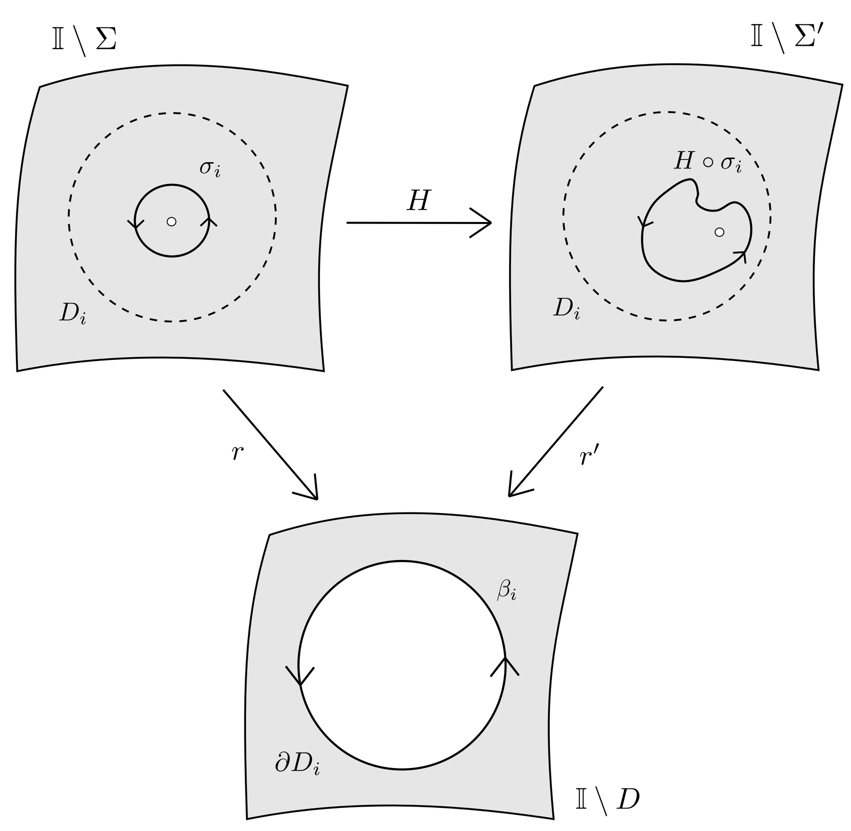

Figure 2

Relationship Between Φ and Ψ

We know by Hurewicz Theorem that $H_1(\mathbb{I} \setminus \Sigma; \mathbb{Z})$ is naturally isomorphic to the abelianization of $\pi_1(\mathbb{I} \setminus \Sigma, b)$. Let $q \colon \pi_1(\mathbb{I} \setminus D, b) \to H_1(\mathbb{I} \setminus D; \mathbb{Z})$ be the canonical projection. Through q every automorphism on $\pi_1(\mathbb{I} \setminus D, b)$ descends to a unique automorphism on $H_1(\mathbb{I} \setminus D; \mathbb{Z})$. This assignment gives raise to a natural and surjective homomorphism $Q \colon \operatorname{Aut}(\pi_1(\mathbb{I} \setminus D, b)) \to \operatorname{Aut}(H_1(\mathbb{I} \setminus D; \mathbb{Z}))$. The following diagram commutes:

$$\begin{array}{ccc} \pi_1(\mathbb{I} \setminus D, b) & \xrightarrow{\Phi(H)} & \pi_1(\mathbb{I} \setminus D, b) \\ \downarrow q & & \downarrow q \\ H_1(\mathbb{I} \setminus D; \mathbb{Z}) & \xrightarrow[\Psi(H)]{} & H_1(\mathbb{I} \setminus D; \mathbb{Z}) \end{array}$$

Lemma. *The kernel of the homomorphism Q is exactly the group of inner automorphisms on $\pi_1(\mathbb{I} \setminus D, b)$ and so for any foliation $\widetilde{\mathcal{F}} \in \Omega$ topologically equivalent to $\mathcal{F}$ and for any conjugating homeomorphism $\mathcal{H}$ the automorphism $\Phi(H)$ is an inner automorphism.*

Conclusions

The above argument proves the Key Lemma.

The Main Result can now be easily deduced from the work of Yu. Ilyashenko [2], [4].

In [6] it is proved a slightly stronger result than the Key Lemma, namely

Theorem. *Let $\mathcal{F}, \widetilde{\mathcal{F}}$ be generic foliations from $\mathcal{A}_2$ which are topologically conjugated and sufficiently close. Then there exists a homeomorphism $\mathcal{H}_0$ that conjugates $\mathcal{F}$ with $\widetilde{\mathcal{F}}$ and has a trivial action on the fundamental group.*

References

[1] X. Gómez-Mont and L. Ortiz. *Sistemas Dinámicos Holomorfos en Superficies.* Aportaciones Matemáticas, Notas de Investigación (Sociedad Matemática Mexicana), 2nd edition, 2004.

[2] Yu. Ilyashenko. The topology of phase portraits of analytic differential equations in the complex projective plane. *Trudy. Sem. Im. I.G.Petrovsk. (1978), vol.4, p.83-136, English transl. Selecta Math. Sov. (1986), vol.5, p.141-199.*

[3] Yu. Ilyashenko and V. Moldavskis. Total rigidity of generic quadratic vector fields. *Moscow Mathematical Journal (2011), vol.11, no.3, p.521-530.*

[4] Yu. Ilyashenko and S. Yakovenko. *Lectures on Analytic Differential Equations.* American Mathematical Society, GSM, 2007.

[5] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial Group Theory: Presentation of Groups in Terms of Generators and Relations.* Dover Publications, Inc., 2nd edition, 1976.

[6] V. Ramírez. Strong topological invariance of the monodromy group at infinity for quadratic vector fields. *In preparation.*

[7] A. A. Shcherbakov. The dynamics of local groups of conformal mappings and the generic properties of differential equations on $\mathbb{C}^2$. *Tr. Mat. Inst. Steklova 254 (2006), Nelinein. Anal. Differ. Uravan., p.111-129, English transl. Proc. Steklov Inst. Math. 3(254) (2006), p.103-120.*