An example of a non-algebraizable singularity

Valente Ramírez
Institut de Recherche Mathématique de Rennes (IRMAR)

Abstract

In this poster we exhibit the first explicit example of a non-algebraizable singularity. This example was constructed in [Ram16].

Introduction

Definition. Let \( F \) be the germ of a holomorphic foliation on \( (\mathbb{C}^2, 0) \) with an isolated singularity. We say that \( F \) is algebraizable (or algebraic-like) if there exist a complex projective surface \( S \) and a point \( p \) on it such that \( F \) is locally holomorphically conjugate to the germ at \( p \) of a globally defined foliation on \( S \).

The existence of non-algebraizable singularities was discovered by Genzmer and Teyssier in [GT10], where they prove the existence of countably many classes of saddle-node singularities which are not algebraizable. Their proof however, does not provide us with any concrete examples of such singularities.

Following the problem suggested in [Cas13], we have constructed an explicit example of the germ of a holomorphic foliation on \( (\mathbb{C}^2, 0) \) which is non-algebraizable.

Transcendence degree of a singularity

We will be able to guarantee that a foliation is not algebraizable if we can guarantee that it has “enough transcendence” encoded into its formal or analytic invariants.

Definition. Let \( \eta = \alpha(x,y) \, dx + \beta(x,y) \, dy \) be a 1-form on \( (\mathbb{C}^2, 0) \). Denote by \( \mathbb{Q}(\eta) \) the field extension of \( \mathbb{Q} \) obtained by adjoining to \( \mathbb{Q} \) the coefficients of the power series expansions of \( \alpha \) and \( \beta \).

Note that this number is not invariant under local changes of coordinates.

In order to make it invariant, we define the transcendence degree of \( \eta \) to be

\[
\text{tr.deg} (\eta) = \min \{ \text{tr.deg} (\mathbb{Q}(\eta)) / \mathbb{Q} \mid \eta \text{ is formally conjugate to } \eta' \},
\]

where \( \text{tr.deg} (\mathbb{Q}(\eta))/\mathbb{Q} \) denotes the transcendence degree of the field extension \( \mathbb{Q}(\eta)/\mathbb{Q} \).

By definition, \( \text{tr.deg} (\eta) \in \mathbb{N} \cup \{ \infty \} \) is an analytic invariant of \( \eta \).

Remark. A polynomial 1-form has finite transcendence degree. Thus any algebraizable singularity has finite transcendence degree.

Our example

Note that our objective is to construct a 1-form \( \omega \) that satisfies \( \text{tr.deg} (\omega) = \infty \).

Theorem. The following form defines a non-algebraizable germ of a holomorphic foliation.

\[
\omega = f_1 f_2 f_3 \sum_{j=1}^3 \lambda_j \frac{df_j}{f_j} + \left( \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!} \right) (x \, dy - y \, dx),
\]

where

\[
\begin{align*}
& f_1 = x, \quad f_2 = y, \quad f_3 = y - x, \\
& \lambda_1 = \pi, \quad \lambda_2 = \sqrt{3}, \quad \lambda_3 = 1 - \lambda_1 - \lambda_2.
\end{align*}
\]

Remark. Note that \( \omega \) is written as

\[
\omega = \omega_0 + x^2 b(x) \eta \rho,
\]

- \( \omega_0 \) is a quadratic homogeneous 1-form,
- \( b(x) \in \mathbb{C}[x] \) is a holomorphic (actually entire) function on \( \mathbb{R} \),
- \( \eta \rho \) is the radial form \( \eta \rho = x \, dy - y \, dx \).

This form defines a non-dicritic degenerate singularity of order (i.e. algebraic multiplicity) two, with separatrix determined by the lines \( f_1 = 0, f_2 = 0, f_3 = 0 \), and non-rational residues \( \lambda_1, \lambda_2, \lambda_3 \).

Normal forms for generic non-dicritic singularities

The formal classification of generic non-dicritic singularities was obtained in [ORV12]. Here we only state the case where the order of the singularity is two.

Theorem ([ORV12]). A generic non-dicritic 1-form \( \eta \) on \( (\mathbb{C}^2, 0) \) having a degenerate singularity of order two is formally equivalent to a formal 1-form \( H \) of the form

\[
H = \eta_3 + x^2 b(x) \eta_1,
\]

where \( \eta_3 \) is the quadratic homogeneous part of \( \eta_3 \), \( b(x) \in \mathbb{C}[x] \), and \( \eta_1 \) is the radial 1-form. Such normal form is unique up to pull-backs by homotheties and multiplication by a scalar factor.

Why does this example work?

- The 1-form \( \omega \) in the example is in its formal normal form.
- The transcendence degree is minimized when a form is in its normal form.
- Indeed, the normalizing map \( \Phi \) taking a form \( \eta \) into its normal form \( \Omega \) is defined by a power series having coefficients in the field \( \mathbb{Q}(\eta) \), and so \( \mathbb{Q}(\Omega) \subset \mathbb{Q}(\eta) \).
- Therefore we have that

\[
\text{tr.deg} (\omega) = \text{tr.deg} (\mathbb{Q}(\omega)/\mathbb{Q}).
\]

- The field \( \mathbb{Q}(\omega) = \mathbb{Q}(x^2, e^{x^3}, e^{x^4}, \ldots) \) has infinite transcendence degree over \( \mathbb{Q} \).
- We conclude that \( \text{tr.deg} (\omega) = \infty \).

The details may be found in [Ram16].

References


Acknowledgements

This result was obtained together with Frank Loray during a short visit to IRMAR during the summer of 2014. Such visit was funded by Laboratorio Internacional Solomon Lefschetz (LADISLA) associated to CNRS (France) and CONACYT (Mexico) and the grant UNAM-DGAPA-PAPIIT IN102413 (Mexico).