An example of a non-algebraizable singularity

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Abstract

In this poster we exhibit the first explicit example of a non-algebraizable singularity. This example was constructed in [Ram16].

Introduction

Definition. Let $\mathcal{F}$ be the germ of a holomorphic foliation on $(\mathbb{C}^2, 0)$ with an isolated singularity. We say that $\mathcal{F}$ is *algebraizable* (or *algebraic-like*) if there exist a complex projective surface S and a point p on it such that $\mathcal{F}$ is locally holomorphically conjugate to the germ at p of a globally defined foliation on S.

The existence of non-algebraizable singularities was discovered by Genzmer and Teyssier in [GT10], where they prove the existence of countably many classes of saddle-node singularities which are not algebraizable. Their proof however, does not provide us with any concrete examples of such singularities.

Following the problem suggested in [Cas13], we have constructed an explicit example of the germ of a holomorphic foliation on $(\mathbb{C}^2, 0)$ which is non-algebraizable.

Transcendence degree of a singularity

We will be able to guarantee that a foliation is not algebraizable if we can guarantee that it has "*enough transcendence*" encoded into its formal or analytic invariants.

Definition. Let $\eta = \alpha(x,y)\,dx + \beta(x,y)\,dy$ be a 1-form on $(\mathbb{C}^2, 0)$. Denote by $\mathbb{Q}(\eta)$ the field extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ the coefficients of the power series expansions of α and β.

Note that this number *is not* invariant under local changes of coordinates.

In order to make it invariant, we define the *transendence degree* of η to be

$$\operatorname{tr.deg}(\eta) = \min\{\operatorname{tr.deg}\,\mathbb{Q}(\tilde{\eta})/\mathbb{Q} \mid \tilde{\eta} \text{ is formally conjugate to } \eta\},$$

where $\operatorname{tr.deg}\,\mathbb{Q}(\tilde{\eta})/\mathbb{Q}$ denotes the transcendence degree of the field extension $\mathbb{Q}(\tilde{\eta})/\mathbb{Q}$.

By definition, $\operatorname{tr.deg}(\eta) \in \mathbb{N} \cup \{\infty\}$ is an analytic invariant of η.

Remark. A polynomial 1-form has finite transcendence degree. Thus *any* algebraizable singularity has finite transcendence degree.

Our example

Note that our objective is to construct a 1-form ω that satisfies $\operatorname{tr.deg}(\omega) = \infty$.

Theorem. *The following form defines a non-algebraizable germ of a holomorphic foliation.*

$$\omega = f_1 f_2 f_3 \sum_{j=1}^{3} \lambda_j \frac{df_j}{f_j} + \left(\sum_{k=0}^{\infty} \frac{x^{k+2}}{e^{k\sqrt{k}}} \right) (x\,dy - y\,dx),$$

where

$$f_1 = x, \quad f_2 = y, \quad f_3 = y - x,$$
$$\lambda_1 = \pi, \quad \lambda_2 = \sqrt{2}, \quad \lambda_3 = 1 - \lambda_1 - \lambda_2.$$

Remark. Note that ω is written as

$$\omega = \omega_0 + x^2 b(x)\,\eta_R,$$

- ω_0 is a quadratic homogeneous 1-form,
- $b(x) \in \mathbb{C}\{x\}$, is a holomorphic (actually entire) function on x,
- η_R is the radial form $\eta_R = x\,dy - y\,dx$.

This form defines a *non-dicritic* degenerate singularity of order (i.e. algebraic multiplicity) two, with separatrices defined by the lines $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, and non-rational residues $\lambda_1, \lambda_2, \lambda_3$.

Normal forms for generic non-dicritic singularities

The formal classification of generic non-dicritic singularities was obtained in [ORV12]. Here we only state the case where the order of the singularity is two.

Theorem ([ORV12])**.** *A generic non-dicritic 1-form η on $(\mathbb{C}^2, 0)$ having a degenerate singularity of order two is formally equivalent to a formal 1-form H of the form*

$$\mathrm{H} = \eta_0 + x^2 b(x) \eta_R,$$

where η_0 is the quadratic homogeneous part of η, $b(x) \in \mathbb{C}[[x]]$ and η_R is the radial 1-form. Such normal form is unique up to pull-backs by homotheties and multiplication by a scalar factor.

Why does this example work?

- The 1-form ω in the example is in its *formal normal form*.
- The transcendence degree is minimized when a form is in its normal form.
- Indeed, the normalizing map Φ taking a form η into its normal form H is defined by a power series having coefficients in the field $\mathbb{Q}(\eta)$, and so $\mathbb{Q}(\mathrm{H}) \subset \mathbb{Q}(\eta)$.
- Therefore we have that

$$\operatorname{tr.deg}(\omega) = \operatorname{tr.deg}\,\mathbb{Q}(\omega)/\mathbb{Q}.$$

- The field $\mathbb{Q}(\omega) = \mathbb{Q}(e^{2\sqrt{2}}, e^{3\sqrt{3}}, \ldots)$ has *infinite transcendence degree* over $\mathbb{Q}$.
- We conclude that $\operatorname{tr.deg}(\omega) = \infty$.

The details may be found in [Ram16].

References

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