Utmost Topological Rigidity for Generic Quadratic Foliations on $\mathbb{CP}^2$

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Abstract

In this poster we consider holomorphic foliations of $\mathbb{CP}^2$ which in a fixed affine chart are induced by a quadratic vector field. In the generic case these foliations have isolated singularities and an invariant line at infinity.

The object of this poster is to present the following result: In the generic case two such foliations may be topologically equivalent if and only if they are conjugate by an affine map on $\mathbb{C}^2$.

This improves, in the case of quadratic foliations, the well–known result of *absolute rigidity* by Ilyashenko [1] which claims that two generic and topologically equivalent polynomial foliations are affine equivalent provided they are close enough in the space of foliations and the linking homeomorphism is close enough to the identity map. Our new result shows that for quadratic foliations all additional hypothesis may be dropped.

This shows for the first time that the paradigm of topological rigidity of polynomial foliations may be formalized: if two generic quadratic foliations are topologically equivalent then they are analytically equivalent.

Introduction

Any polynomial vector field on $\mathbb{C}^2$ with isolated singularities defines a singular holomorphic foliation which can be analytically extended to $\mathbb{CP}^2$. Define $\mathcal{A}_n$ to be the class of singular foliations on $\mathbb{CP}^2$ which in a *fixed* affine chart are induced by a polynomial vector field of degree n. Generically, these foliations have an invariant line at infinity carrying $n+1$ singularities.

Polynomial foliations exhibit a phenomena known as topological rigidity, which was until now understood to be a heuristic idea rather than a theorem. The idea of topological rigidity is that the topological equivalence of generic foliations implies their analytic equivalence. The first rigidity property was discovered by Ilyashenko and was called *absolute rigidity*.

Theorem ([1]). *A generic foliation $\mathcal{F} \in \mathcal{A}_n$ is absolutely rigid; that is, there exist a neighborhood U of $\mathcal{F}$ in $\mathcal{A}_n$ and a neighborhood V of the identity map in the space of homeomorphisms of $\mathbb{CP}^2$ such that any foliation from U which is conjugate to $\mathcal{F}$ via a homeomorphism in V is affine equivalent to $\mathcal{F}$.*

Later Ilyashenko and Moldavskis proved a stronger property for quadratic foliations known as *total rigidity*.

Theorem ([2]). *A generic foliation $\mathcal{F} \in \mathcal{A}_2$ is totally rigid; namely, there exist but a finite number of foliations from the class $\mathcal{A}_2$ which are topologically equivalent to $\mathcal{F}$ modulo affine equivalence.*

Our main result shows that in fact two generic quadratic foliations may be topologically equivalent if and only if they are affine equivalent.

A convention

Generic quadratic foliations have three singular points on the line at infinity so, without loss of generality, we may assume all foliations considered have the same singular points at infinity. Under this assumption we define analytic conjugacy of monodromy groups as follows:

Definition. *Two monodromy representations*

$$\Delta^j \colon \pi_1(\mathcal{L}, b) \to \mathrm{Diff}(\mathbb{C}, 0), \quad \gamma \mapsto \Delta^j_\gamma, \qquad j = 1, 2,$$

are analytically conjugate if there exists a germ $h \in \mathrm{Diff}(\mathbb{C}, 0)$ such that

$$h \circ \Delta^1_\gamma = \Delta^2_\gamma \circ h \qquad \forall \gamma \in \pi_1(\mathcal{L}, b).$$

Main results

Theorem 1. *Let $\mathcal{F} \in \mathcal{A}_2$ be a generic foliation and suppose its monodromy group at infinity is analytically conjugate to the monodromy group of some $\widetilde{\mathcal{F}} \in \mathcal{A}_2$. There exists an affine map on $\mathbb{C}^2$ that conjugates $\mathcal{F}$ and $\widetilde{\mathcal{F}}$.*

Corollary 1. *Let $\mathcal{F}, \widetilde{\mathcal{F}} \in \mathcal{A}_2$ be generic foliations. The following are equivalent:*

1. There exists an affine map on $\mathbb{C}^2$ conjugating $\mathcal{F}$ and $\widetilde{\mathcal{F}}$,

2. There exists a homeomorphism on $\mathbb{CP}^2$ conjugating $\mathcal{F}$ and $\widetilde{\mathcal{F}}$,

3. Foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ have analytically conjugate monodromy groups.

Remark. The implication 2. $\Rightarrow$ 3. using *our convention* of analytic conjugacy of monodromy groups is the main result of [5].

Genericity assumptions

In the above theorem we assume foliation $\mathcal{F}$ satisfies the following conditions:

(i) The monodromy group at infinity is non–solvable,

(ii) The characteristic numbers $\lambda_1, \lambda_2, \lambda_3$ of the singularities at infinity are pairwise different and do not belong to the set $\frac{1}{3}\mathbb{Z} \cup \frac{1}{4}\mathbb{Z} \cup \frac{1}{5}\mathbb{Z}$,

(iii) The commutator of the monodromy maps corresponding to geometric generators of the fundamental group of the infinite leaf is a parabolic germ with non–trivial quadratic part.

Moreover, there is an additional technical assumption. In the proof we construct a complex Zariski open set $\mathcal{U} \subset \mathcal{A}_2$ and assume

(iv) Foliation $\mathcal{F}$ belongs to the set $\mathcal{U}$.

The proof

A generic foliation is induced in a neighborhood of the line at infinity by a rational differential equation

$$\frac{dz}{dw} = \frac{z\,P(z, w)}{Q(z, w)} = F(z, w). \tag{1}$$

After a normalization process introduced in [3] we may assume that the right hand side of equation (1) is given in terms of five complex parameters: the characteristic numbers λ_1, λ_2 and three more parameters $\alpha_0, \alpha_1, \alpha_2$. We write

$$\mathcal{F}(\lambda, \alpha)$$

to denote the foliation induced by (1) with parameters $\lambda = (\lambda_1, \lambda_2)$ and $\alpha = (\alpha_0, \alpha_1, \alpha_2)$.

Suppose that $\mathcal{F}(\lambda, \beta)$ has its monodromy group at infinity analytically conjugate to that of $\mathcal{F}(\lambda, \alpha)$. The idea of the proof is to compute the first terms in the power series expansion of some *distinguished parabolic elements* of the monodromy group of $\mathcal{F}(\lambda, \beta)$ in terms of λ and β and deduce that such conjugacy of the monodromy groups imposes several *polynomial* conditions on the parameters β. These computations are inspired by and follow closely those presented in [4].

Theorem 1 is proved by the following series of lemmas.

The key lemma

Consider equation (1) and let

$$\frac{s(w)}{r(w)} = \frac{d}{dz}\Big|_{z=0} F(z, w),$$

with s, r relatively prime polynomials and r monic. Let also

$$\varphi(w) = \exp\left(\int_b^w \frac{s(\tau)}{r(\tau)}\, d\tau\right)$$

be the first variation of equation (1) with respect to the solution $z \equiv 0$.

Let μ_1, μ_2 be standard geometric generators of $\pi_1(\mathcal{L}, b)$ and define

$$\gamma_* = \mu_2 \mu_1 \mu_2^{-1} \mu_1^{-1}.$$

Lemma 1. *For every $d = 3, 4, 5, 6$ there exists a polynomial $P_d(w)$ such that the existence of a germ $h \in \mathrm{Diff}(\mathbb{C}, 0)$ that conjugates the monodromy groups of $\mathcal{F}(\lambda, \alpha)$ and $\mathcal{F}(\lambda, \beta)$ up to jets of order d implies*

$$\int_{\gamma_*} \frac{P_d(w)}{r(w)^d}\, \varphi(w)^{d-1}\, dw = 0,$$

where r, φ and γ_* are as defined above.

Remark. The coefficients of the polynomials $P_d(w)$ depend polynomially on the parameter $\beta = (\beta_0, \beta_1, \beta_2)$. Also they depend polynomially on α and rationally on λ.

Seccond lemma

Lemma 2. *There exist polynomials $F_d(\beta_0, \beta_1, \beta_2)$, $d = 3, \ldots, 6$, such that if the polynomials $P_d(w)$ defined in the Key lemma satisfy*

$$\int_{\gamma_*} \frac{P_d(w)}{r(w)^d}\, \varphi(w)^{d-1}\, dw = 0,$$

then

$$F_d(\beta_0, \beta_1, \beta_2) = 0.$$

Deducing this second lemma from the Key lemma follows easily from a result in [4].

Elimination lemma

Lemma 3. *There exists a dense Zariski open set $U \subset \mathbb{C}^5$ such that the system of polynomial equations*

$$F_3(\beta_0, \beta_1, \beta_2) = 0, \; \ldots, \; F_6(\beta_0, \beta_1, \beta_2) = 0$$

has a unique solution given by $(\beta_0, \beta_1, \beta_2) = (\alpha_0, \alpha_1, \alpha_2)$, provided $(\lambda, \alpha) \in U$.

This lemma is proved taking successive resultants of the polynomials F_d with respect to $\beta_2, \beta_1, \beta_0$.

Summary of the proof

The main steps of the proof may be sumarized as follows:

- We normalize quadratic foliations using the action of the affine group of $\mathbb{C}^2$ following [3].
- We consider two generic and normalized foliations $\mathcal{F}(\lambda, \alpha)$ and $\mathcal{F}(\lambda, \beta)$ with conjugate monodromy groups.
- We compute explicit expressions for *certain* parabolic germs in the monodromy groups in terms of the parameters λ, α, β.
- We deduce that such conjugacy of the monodromy groups imposes several *non–polynomial* conditions on the parameter β.
- We then translate these conditions into *polynomial equations* via Lemma 2.
- We show that if the parameters $(\lambda, \alpha) \in \mathbb{C}^5$ are generic then such system of equations has a unique solution $\beta = \alpha$.
- This shows that any two generic and normalized foliations with conjugate monodromy groups are one and the same.
- In particular two generic foliations, not necessarily normalized, with conjugate monodromy groups are affine equivalent.

References

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