On fixed-point theorems and self-maps of projective spaces

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Self-maps of projective space

Consider $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^{n+1})$ the **complex projective space** of dimension n.

More precisely,

Definition

We define $\mathbb{P}^n_{\mathbb{C}}$ to be the space of lines through the origin in $\mathbb{C}^{n+1}$:

$$\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

$$(x_0, x_1, \ldots, x_n) \sim (\lambda x_0, \lambda x_1, \ldots, \lambda x_n) \text{ for every } \lambda \in \mathbb{C}^*.$$

The equivalence classes of $\sim$, denoted $[x_0 : x_1 : \ldots : x_n]$, are called **homogeneous coordinates**.

Self-maps of projective space

A **regular self-map** (or holomorphic self-map) of $\mathbb{P}^n_{\mathbb{C}}$ is a polynomial map $f\colon \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ given by

$$[x_0 : \ldots : x_n] \longmapsto [P_0(x_0, \ldots, x_n) : \ldots : P_n(x_0, \ldots, x_n)],$$

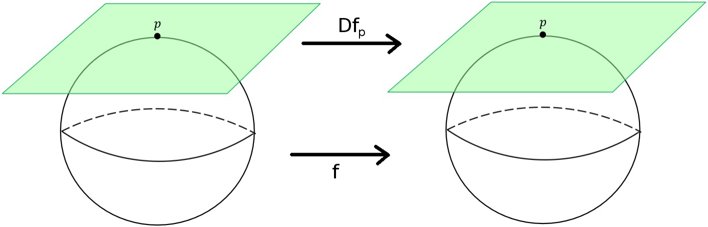
where

- All P_i are homogeneous polynomials of the same degree,
- There is no point in which all the P_i vanish simultaneously.

The common degree of the P_i is called the (algebraic) **degree** of f.

Self-maps of projective space

If p is a fixed-point of f, then the derivative map Df_p maps the tangent space $T_p\mathbb{P}^n_{\mathbb{C}}$ to itself.



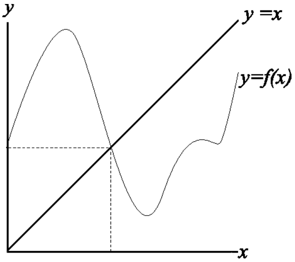
The **eigenvalues** of the derivative map are called the **multipliers** of f at p.

They give important information about the **local behavior** of f around p.

Self-maps of projective space

Definition

We say that a fixed point p is **non-degenerate** if $\det(\mathrm{I} - \mathrm{D}f_p) \neq 0$.



Non-degenerate fixed points are those where the **graph** of f intersects the **diagonal** transversally.

A map is called **transversal** if it all its fixed points are isolated and non-degenerate.

Self-maps of projective space

Let us group together all self-maps of degree d on $\mathbb{P}^n_{\mathbb{C}}$ and call $\mathrm{End}(n, d)$ the space of all these.

Remark: We're only interested in the case $d > 1$.

Proposition

A *typical* representative $f \in \mathrm{End}(n, d)$ has exactly

$$N(n, d) := 1 + d + d^2 + \ldots + d^n = \frac{d^{n+1} - 1}{d - 1}$$

fixed points, all of then non-degenerate.

Self-maps of projective space

Thus, a typical element of $\text{End}(n, d)$ has N fixed points, each defining n multipliers.

We obtain a collection of nN complex numbers that depend our choice of $f \in \text{End}(n, d)$.

Objective:

Understand this collection as a global property of f.

Some questions we can ask:

- Which collections of numbers may be realized as the set of multipliers of a self-map?
- Which maps share the same set of multipliers?
- If we know the set of multipliers, can we recover the map?

Fixed-point theorems and multipliers

Interlude: Fixed-point theorems

When does a self-map have a fixed point?

The problem

Let X be a compact oriented manifold and $f : X \to X$ a continuous map. When can we guarantee that f has a fixed point?

The strategy

As pointed out before, a fixed point is a point of intersection between the graph of f and the diagonal Δ, i.e.

$$\mathrm{Fix}(f) = \Gamma_f \cap \Delta \subset X \times X.$$

The intersection number $\#(\Gamma_f \cap \Delta)$ depends only on the homology classes of Γ_f and Δ.

We should be able to detect whether or not Γ_f and Δ intersect by using homology/cohomology theory!

The Lefschetz fixed-point theorem

The theorem

Define the **Lefschetz number** of f to be

$$L(f) = \sum_k (-1)^k \operatorname{tr}(f^*\colon H^k(X,\mathbb{Q}) \to H^k(X,\mathbb{Q})).$$

Theorem: *If $L(f) \neq 0$ then f has a fixed point.*

Theorem (Lefschetz fixed-point formula)

Assume f has isolated fixed points only. Then f has exactly $L(f)$ fixed points (counted with multiplicity).

The Lefschetz fixed-point theorem

Some immediate corollaries

- Brouwer's fixed point theorem.
- Every self-map of a contractible manifold has a fixed point.
- Every self-map of a $\mathbb{Q}$-acyclic manifold has a fixed point.
- A map $f \in \mathrm{End}(n, d)$ has exactly $N(n, d)$ fixed points.

The global topology of X and the way f^* acts on $H^\bullet(X, \mathbb{Q})$, constrain the **quantity** of fixed points.

The Lefschetz formula is very powerful, but it doesn't tell us much about the local **behavior** of the fixed points.

The Lefschetz fixed-point theorem

Question:

Can we improve Lefschetz' theorem to tell us something about the **behavior** (ie. about the multipliers) of f at its fixed points?

For example, what if we use $H^\bullet_{dR}(X)$ instead?

Answer:

No. The ring $H^\bullet_{dR}(X) \cong H^\bullet(X, \mathbb{R})$ cannot detect the local behavior of f around the fixed points.

We need to assume extra structure on X and f.

The cohomology of complex manifolds

The complex case

Let us now consider X a compact complex manifold and $f: X \to X$ a holomorphic map.

Complex manifolds have richer cohomology. They are equipped with **Dolbeault cohomology groups** $H^{p,q}_{\bar\partial}(X) \cong H^q(X, \Omega^p_X)$ which in some sense refine the de Rham cohomology.

For Kähler manifolds (including all submanifols of $\mathbb{P}^n_{\mathbb{C}}$) their relation is quite straightforward:

Hodge decomposition

$$H^k_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X).$$

The holomorphic Lefschetz fixed-point theorem

Definition

The *holomorphic Lefschetz number* of f is defined to be

$$L(f, \mathcal{O}_X) = \sum_{q=0}^{n} (-1)^q \operatorname{tr}\left(f^* \colon H_{\bar{\partial}}^{0,q}(X) \to H_{\bar{\partial}}^{0,q}(X) \right).$$

Theorem (Holomorphic Lefschetz fixed-point theorem)

If f has only non-degenerate fixed points then

$$\sum_{x \in \operatorname{Fix}(f)} \frac{1}{\det(\mathrm{I} - \mathrm{D}f_x)} = L(f, \mathcal{O}_X).$$

The holomorphic Lefschetz fixed-point theorem

An example

Let $X = \mathbb{P}^1_{\mathbb{C}}$ be the Riemann sphere and $f\colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be a rotation given in some affine chart $\mathbb{C} \subset \mathbb{P}^1_{\mathbb{C}}$ as

$$f(z) = e^{i\theta} z.$$

The topological Lefschetz number is $L(f) = 2$.
Indeed, this map has two fixed points, given by $z = 0$ and $z = \infty$.

The holomorphic Lefschetz number is $L(f, \mathcal{O}) = 1$.

Let λ be the multiplier of f at $z = \infty$. Then λ satisfies

$$\frac{1}{1 - e^{i\theta}} + \frac{1}{1 - \lambda} = 1.$$

This implies that $\lambda = e^{-i\theta}$.

The holomorphic Lefschetz fixed-point theorem

An example

Let us verify this prediction.

In coordinates $w = \frac{1}{z}$, the map f is given by

$$w \mapsto \frac{1}{e^{i\theta} \cdot \frac{1}{w}} = e^{-i\theta} w,$$

and we immediately see that $\lambda = Df_{w=0} = e^{-i\theta}$.

The holomorphic Lefschetz fixed-point theorem

And beyond...

The holomorphic Lefschetz fixed-point theorem is a particular case of a very general theorem called the **Woods Hole fixed-point theorem** (or Atiyah-Bott fixed point theorem).

Back to self-maps of projective space

Using the Woods Hole formula we can prove the following:

Theorem

Let $\phi\colon \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}$ be a polynomial symmetric function of degree at most n, and let $f \in \mathrm{End}(n,d)$ be a transversal self-map of $\mathbb{P}^n_{\mathbb{C}}$. Then

$$\sum_{p \in \mathrm{Fix}(p)} \frac{\phi(\mathrm{D}f_p)}{\det(\mathrm{I} - \mathrm{D}f_p)}$$

is a constant that only depends on n, d and ϕ.

This provides several **fixed-point theorems**!

The relations for self-maps of projective space

For $\mathbb{P}^1_{\mathbb{C}}$ this only recovers the holomorphic Lefschetz formula.

For self-maps of $\mathbb{P}^2_{\mathbb{C}}$ we have the following relations:

- $$\sum_{p \in \mathrm{Fix}(p)} \frac{1}{\det(\mathrm{I} - \mathrm{D}f_p)} = 1,$$
- $$\sum_{p \in \mathrm{Fix}(p)} \frac{\mathrm{tr}(\mathrm{D}f_p)}{\det(\mathrm{I} - \mathrm{D}f_p)} = -d,$$
- $$\sum_{p \in \mathrm{Fix}(p)} \frac{\mathrm{tr}(\mathrm{D}f_p)^2}{\det(\mathrm{I} - \mathrm{D}f_p)} = d^2.$$

The relations for self-maps of projective space

Question:

Do the above equations generate **all** relations among the multipliers?

Answer:

No. We know that many more equations **must exist**, but we do not know them!

Remark: From now on we will focus on the smallest interesting case: degree 2 maps on $\mathbb{P}^2_{\mathbb{C}}$.

The relations for self-maps of projective space

Why do we know more equations exist?

The multiplier map for $n = d = 2$

The assignment

$$f \longmapsto \{\text{multipliers of } f\}$$

defines a rational map

$$\mathcal{M}\colon \operatorname{End}(2,2)/\operatorname{PGL}(2,\mathbb{C}) \dashrightarrow (\mathbb{C}^2)^7/S_7.$$

- The fibers are finite,
- The dimension of the domain is 9,
- The codimension of the closure of the image is 5.

This means that there exist at least **5 independent equations** among the multipliers (but **we only know three!**).

The relations for self-maps of projective space

The big question

What are the missing relations?

The relations for self-maps of projective space

A very particular case...

Together with Adolfo Guillot, we have constructed relations for the subfamily of End(2, 2) having an **invariant line** ie. $f(\ell) \subset \ell$.

The new relations for our particular case

Consider quadratic self-maps with an invariant line. Let p_1, p_2, p_3 be the fixed points on the line and p_4, p_5, p_6, p_7 the fixed points away from the line.

Denote by u_i, v_i the multipliers of f at p_i.

Theorem

For any rational symmetric function $\varphi \in \mathbb{C}(u_1, v_1, \ldots, u_3, v_3)$ *there exist polynomials* $A_k \in \mathbb{C}[u_4, v_4, \ldots, u_7, v_7]$, $k = 1, \ldots, 4$, *such that*

$$A_0 + A_1\varphi + A_2\varphi^2 + A_3\varphi^3 + A_4\varphi^4 = 0,$$

when evaluated at the multipliers of any $f \in \mathrm{End}(2, 2)$ *with an invariant line.*

This actually gives all the missing relations.

The new relations for our particular case

These equations are extremely complicated!

It follows from a previous collaboration with Yury Kudryashov that, in general, these relations cannot be rewritten in the form

$$\sum_{p \in \mathrm{Fix}(p)} \phi(\mathrm{D}f_p) = C_\phi$$

for any **rational** invariant function $\phi\colon \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{C}$.

These relations **do not come from a fixed-point theorem** as before.

The new relations for our particular case

But $\mathrm{End}(2, 2)$ with an invariant line is just a very particular case, there is still a lot of work to do...

Thank you!

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