Differential Equations on the Complex Plane

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Introduction

Let us consider the following differential equation

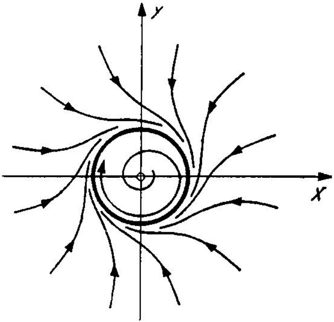
$$\frac{dx}{dt} = Q(x,y) \quad \frac{dy}{dt} = P(x,y), \tag{1}$$

where $(x,y) \in \mathbb{R}^2$ and P, Q are real polynomials.

Introduction

For example,

$$\begin{aligned} \dot{x} &= x + y - x^3 + xy^2 \\ \dot{y} &= -x + y - x^2y + y^3 \end{aligned} \tag{2}$$



Phase portrait of equation (2)

Introduction

It is convenient to identify ODEs with vector fields.
In this way, equation (1) defines an analyitic foliation of the phase space $\mathbb{R}^2$ outside the singular locus

$$\Sigma = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; P(x, y) = Q(x, y) = 0 \right\}.$$

We are interested in studying the topology of such foliations.

Topological and analytic equivalence

Definition

Two singular foliations $\mathcal{F}_1$, $\mathcal{F}_2$ in $\mathbb{R}^2$ are called topologically equivalent if there exists a homeomorphism $\mathcal{H}$ that maps the leaves of $\mathcal{F}_1$ onto the leaves of $\mathcal{F}_2$ and defines a bijection between the singular points.

If such homeomorphism is an analytic mapping we say that the above foliations are analytically equivalent.

Topological and analytic equivalence

Suppose $\mathcal{F}$ is the singular foliation induced by equation (1).
A natural question arises:

Question

What happens with the topology of $\mathcal{F}$ when we perturb the coefficients of the polynomials P and Q that define the equation?

This is a fundamental question in applied mathematics!

Answer

Generic planar systems are structurally stable.

Limit sets

Next question

What are the limit sets of equation (1)?

Poincaré-Bendixon Theorem

A compact, connected ω-limit set of a planar system can only be:

- A singular point,
- A limit cycle,
- A finite amount of singular points together with orbits connecting them.

Limit cycles

Theorem

Any planar polynomial system has only a finite amount of limit cycles.

For example, linear systems do not have any limit cycles.

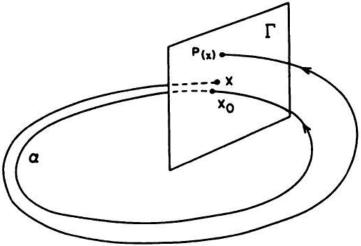
Hilbert's 16th Problem

Let $n \geq 2$. Determine an upper bound for the amount of limit cycles that a planar polynomial system of degree n may have.

- We still don't even known that such a bound exists!

The Poincaré-map

Limits cycles are usually studied via the Poincaré map.



The Poincaré map

It is convenient to think of the Poincaré map as a mapping

$$P\colon \pi_1(\alpha, x_0) \longrightarrow \mathrm{Diff}(\Gamma, x_0).$$

Generic properties of polynomial foliations

In summary

The follwoing properties are generic for polynomial foliations:

- Structural stability
- Finitely many limit cycles
- Leaves may accumulate only to singular points and limit cycles

The Petrovskii-Landis strategy

In 1957 I.G. Petrvskii and E.M. Landis claimed to have a proof of Hilbert's 16th problem.

The strategy:

Consider a planar system

$$\dot{x} = Q(x,y) \quad \dot{y} = P(x,y).$$

- Extend the domain of definition to $(x,y) \in \mathbb{C}^2$.
- Find a bound for the *complex limit cycles* that the equation may have on the complex plane.

There was a crucial mistake on the proof!!
Even though the proof is no good, this opened a door for a fascinating new theory.

Polynomial foliations on $\mathbb{C}^2$

Let us now consider the same equation

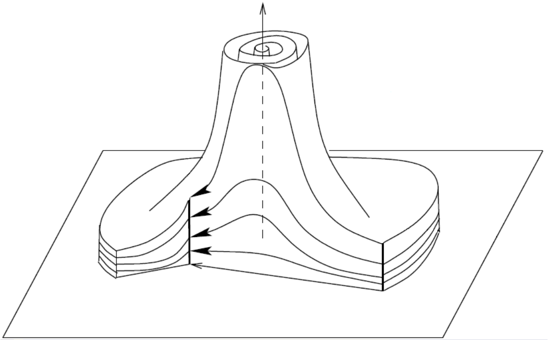
$$\dot{x} = Q(x, y) \quad \dot{y} = P(x, y),$$

but this time with $(x, y) \in \mathbb{C}^2$ and $P, Q \in \mathbb{C}[x, y]$.

- The solutions to the equation are now complex curves immersed into $\mathbb{C}^2$.
- This defines a holomorphic foliation of $\mathbb{C}^2 \setminus \Sigma$ by analytic curves.
- Namely, a foliation by real surfaces of a 4-dimensional real manifold.

Polynomial foliations on $\mathbb{C}^2$

For example, a linear foliation would look something like this:



Extenssion to $\mathbb{C}P^2$

Let us compactify the plane $\mathbb{C}^2$ by adding a *line at infinity*. This gives us the complex projective plane.

- In the new affine coordinates

$$(z, w) = (1/x, y/x)$$

  the line at infinity $\mathbb{I}$ is described by the equation $z = 0$.

- This coordinate change transforms (orbitally) equation (1) into

$$\begin{aligned} \dot{z} &= z\widetilde{Q}(z, w) \\ \dot{w} &= w\widetilde{Q}(z, w) - \widetilde{P}(z, w) \end{aligned} \qquad (3)$$

The monodromy group at infinity

A *generic* polynomial foliation $\mathcal{F}$ has an invariant line at infinity. Thus $\mathcal{L}_{\mathcal{F}} = \mathbb{I} \setminus \mathrm{Sing}(\mathcal{F})$ is a leaf of the foliation.

Let us consider the complex Poincaré map associated to each loop in the infinite leaf.
This gives a map

$$\pi_1(\mathcal{L}_{\mathcal{F}}, z_0) \longrightarrow \mathrm{Diff}(\Gamma, z_0).$$

Its image, G, is the monodromy goup at infinity of foliation $\mathcal{F}$.

The monodromy group at infinity

Topologically equivalent foliations have conjugated monodromy groups.

Suppose $G_1 = \langle f_1, \ldots, f_n \rangle$ and $G_2 = \langle g_1, \ldots, g_n \rangle$ are the monodromy groups of two topologically conjugated foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. There exists a germ h such that for each $i = 1, \ldots, n$

$$h \circ f_i = g_i \circ h,$$

Under some mild extra assumptions we may conclude that h is the germ of a holomorphic mapping.

Generic properties for monodromy groups

The monodromy group G of a generic foliation satisfies:

- G is topologically rigid,
- G has infinitely many elements which have different isolated fixed points,
- The orbit of every point in $\Gamma \setminus \{x_0\}$ is dense in Γ.

Generic properties for complex foliations

The previous properties imply that the foliation $\mathcal{F}$ satisfies

- F is topologically rigid,
- F has infinitely many complex limit cycles,
- Every leaf of $\mathcal{F}$ different from the infinite line is dense in all $\mathbb{C}P^2$.

Conclusions

Some objects that have appeared in this talk:

- Real and complex differential equations
- Continuous and discrete complex dynamics
- Foliations on algebraic varieties
- Algebraic curves and Riemann surfaces
- Homeomorphisms on a punctured sphere, automorphisms of its fundamental group

There are still a lot of problems to be solved!